* * SOLUTIONS * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiners: R. Buchweitz
R. Grinnell

Date: February 14, 2014
Time: 3:00 pm
Duration: 120 minutes

(Print) Surname: * SOLUTIONS *

(Print) Given Name(s): (and basic statistics)

Student Number: ____________________

Signature: ____________________

Circle the name of your Teaching Assistant and Tutorial Number:

Taylor ESCH 2 5

Rui GAO 10 21

Reggie (Zejun) LIU 26

Yishen SONG 9 25

Huiyi WANG 13 23

6 7 Ethan (Junsheng) WU

12 19 Kevin YAN

4 8 Ric (Biyun) ZHANG

15 Elaine (Mengnan) ZHU

Read these instructions:

1. This test has 11 pages. It is your responsibility to check at the beginning of the test that all of these 11 pages are included.
2. If you need extra answer space for any question, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. The following are forbidden at your workspace: calculators, any other kind of electronic aid or device (e.g. cell/smart phones, i-pads, i-phones, etc.), scrap paper, food, textbooks, bags, pencil/pen carrying cases, drinks in paper cups or boxes or similar container that has a removable label.
4. Cell/smart/i-phones phones must be turned off and left at the front of the test room.
5. You are encouraged to write in pen or other ink, not pencil. If any part of the your test is written in pencil, then you will be denied any re-grading privilege.

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6	7
B	A	A	D	D	D	A

Do not write anything in the boxes below.

Info.	Part A
2	21

Part B

1	2	3	4	5	6
16	17	14	11	13	6

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 3 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations and rough work.

1. If $A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = [B_{ij}] = AC - C^{-1}$ then $B_{11} + B_{22}$ is

(A) 34 $\boxed{\text{(B) } 24}$ (C) 25 (D) 26 (E) none of (A) - (D)

$$AC = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 13 & 22 \end{bmatrix} \qquad \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_{11} + B_{22} = 24$$

$$C^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

$$AC - C^{-1} = \begin{bmatrix} 7 & 11 \\ 13 & 22 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 16 \\ 14 & 19 \end{bmatrix} = B$$

2. If A is the matrix in Question 1 above, then the value of $det(2A^T)$ is

$\boxed{\text{(A) } 44}$ (B) 22 (C) -20 (D) -10 (E) none of (A) - (D)

$$\det(2A^T) = 2^2 \det(A) = 4(3+8) = 44$$

OR $2A^T = \begin{bmatrix} 6 & 8 \\ -4 & 2 \end{bmatrix} \Rightarrow \det(2A^T) = 12 + 32 = 44$

3. The minimum value of $Z = x - 3y$ subject to the four constraints

$$x \geq -1 \qquad y \geq 0 \qquad x + 2y \leq 3 \qquad y = x \qquad \text{is}$$

$\boxed{\text{(A) } -2}$ (B) -7 (C) -1 (D) 0 (E) none of (A) - (D)

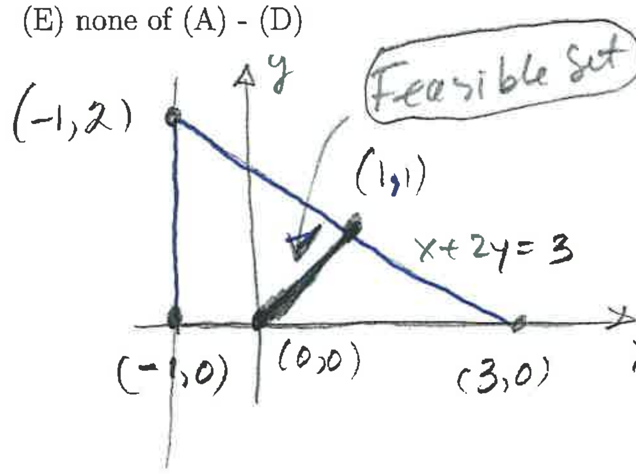
$Z(0,0) = 0$

$Z(1,1) = 1 - 3 = -2$ A-MIN

$Z(-1,2) = -1 - 6 = -7$

$Z(3,0) = 3$

$Z(-1,0) = -1$



Feasible set is only the line segment joining (0,0) to (1,1).

$3x + y = 2/3$
$2x + \frac{2}{3}y = 2$
$0x + 0y = -7/3$

4. For what value(s) of the real constant c is the system of equations $\begin{array}{l} 3x + y = c \\ 2x + cy = 2 \end{array}$ inconsistent?

(A) 3/2 (B) −2/3 (C) 3 $\boxed{\text{(D) } 2/3}$ (E) {(B) and (C)} (F) {(C) and (D)}

$$\left(\begin{array}{cc|c} 3 & 1 & c \\ 2 & c & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 3 & 1 & c \\ 0 & c-\frac{2}{3} & 2-\frac{2}{3}c \end{array}\right) \qquad 2 - \frac{4}{9} = \frac{14}{9}$$

If $c = \frac{2}{3}$ ⇒ last row yields $0x + 0y = \frac{14}{9}$ $\times$

5. A 2×2 matrix B satisfies the equation $3B^2 - 4B = -I$. The inverse of B is

(A) $-3B + 4$ (B) $3B + 4I$ (C) $3B - 4I$ $\boxed{\text{(D) } -3B + 4I}$
(E) a matrix not in (A) - (D) (F) impossible to find because we do not know B exactly.

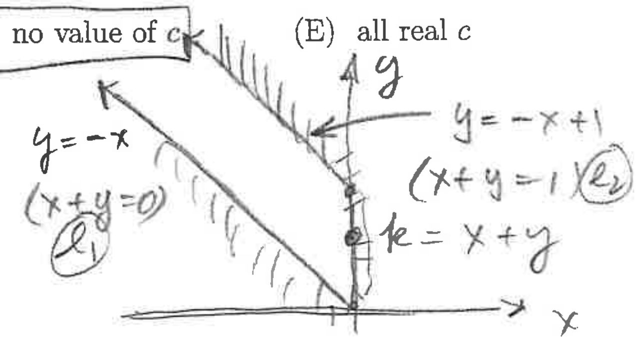
$3B^2 - 4B = -I$
$-3B^2 + 4B = I$
$B(-3B + 4I) = I$

$B^{-1} = -3B + 4I$

6. For what value(s) of the real constant c does the function $Z = cx + cy$ not have a maximum when $0 \le x + y \le 1$ and $x \le 0$?

× (A) 3 × (B) 0 × (C) −1 $\boxed{\text{(D) no value of } c}$ (E) all real c

$Z = c(x+y)$
$= \begin{cases} 0 & \text{if on } \ell_1 \\ c & \text{if on } \ell_2 \end{cases}$



7. Exactly how many of the following statements are always true?

✗ • A linear objective function defined on a non-empty, standard feasible region must have a *minimum or maximum* or both.

✗ • A square matrix is invertible if and only if all of its diagonal entries are nonzero.

✗ • Every system of 2 linear equations in 3 variables is consistent.

✓ • A 3×3 matrix P is equivalent to I_3 if and only if $det(P) \neq 0$

$\boxed{\text{(A) } 1}$ (B) 2 (C) 3 (D) 4 (E) 0

*** DID YOU PUT THE ANSWERS IN THE BOXES ON PAGE 2? ***

Part B - Full Solution Problem Solving Put your answers and solutions in the space provided. Full points will be awarded for your solutions if and only if they are correct, complete, and show sufficient relevant concepts form MATA33.

1. In all of this question let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

   (a) Find A^{-1} by the method of row-reduction. [8 points]

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$-\frac{1}{3} R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1/3 \end{array}\right]$$

$$R_2 + 3R_3 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right]$$

$$\begin{matrix} R_1 - R_3 \to R_1 \\ -1/2\, R_2 \to R_2 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/3 \end{bmatrix}$$

   (b) Use your answer to (a) to solve the equation $AX = B$ where $B = \begin{bmatrix} 2 \\ -4 \\ 12 \end{bmatrix}$ [4 points]

$$X = A^{-1}B = \begin{bmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 7 \\ -2 \end{bmatrix}$$

   (c) Express B as a sum of scalar multiples of the columns of A. [4 points]

$\because X = \begin{bmatrix} 4 \\ 7 \\ -2 \end{bmatrix}$ and $AX = B$ we have

$$\underbrace{4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 7\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{Columns of } A} = \underbrace{\begin{bmatrix} 2 \\ -4 \\ 12 \end{bmatrix}}_{B}$$

2. Find the maximum and minimum values of the function $Z = -10x + 4y$ subject to the five constraints:

$$x \geq -2 \qquad y \geq -3 \qquad \underbrace{-x + y \leq 3}_{l_1} \qquad \underbrace{x + 4y \leq 12}_{l_2} \qquad \underbrace{5x - 2y \leq 16}_{l_3}.$$

You answer must include a neat, labeled feasible region, the location of all optimizing points, and sufficient calculation details. [17 points]

$l_1 \cap l_2$:

$-x + y = 3$
$x + 4y = 12$
$5y = 15$
$\therefore y = 3$
Point $= (0, 3)$

$l_2 \cap l_3$:

$x + 4y = 12$
$5x - 2y = 16$
$11x = 44$
$\therefore x = 4$
Point $= (4, 2)$

l_1 intercepts:

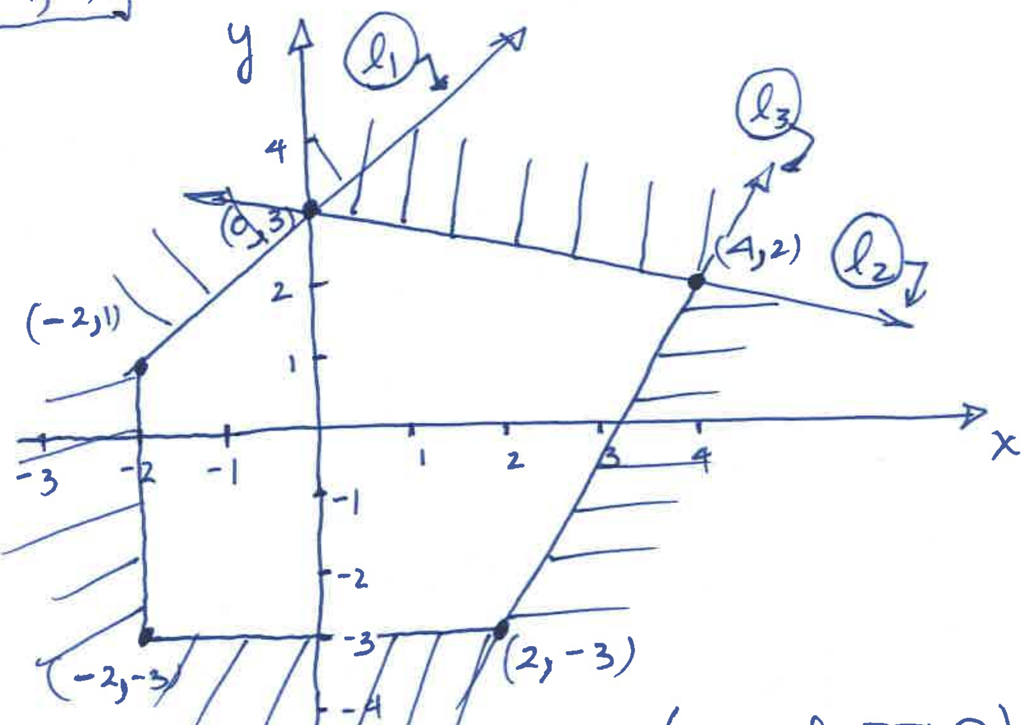
$(0,3), (-3,0)$

l_2 intercepts:

$(12,0), (0,3)$

l_3 intercepts:

$x = \frac{16}{5}$ $\left(\frac{16}{5}, 0\right)$
$y = -8$ $(0, -8)$



S = feasible region.
(0,0) satisfies all five inequalities above $\therefore (0,0) \in S$ and clearly $S \neq \emptyset$

As usual, the "non-S" region is shaded by ////

It is clear that S is bounded.

S has five corner points:
$(-2,-3)$, $(-2,1)$, $(0,3)$, $(4,2)$, $(2,-3)$

By FTLP, optimal values of Z occur at corner points of S

Evaluation (b/c of FTLP):

$Z(-2,-3) = 8$
$Z(-2,1) = 24 \leftarrow$ MAX
$Z(0,3) = 12$
$\left.\begin{array}{l} Z(4,2) = -32 \\ Z(2,-3) = -32 \end{array}\right\}$ MIN

$\boxed{\text{ANSWER (FINAL)}}$:

MAX value of Z is 24 @ (−2, 1).

MIN value of Z is −32 @ all points joining (4,2) to (2,−3).

3. (a) Use the method of reduction to solve the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - x_4 + x_5 &= 4 \\ x_2 + 6x_3 - 3x_4 - x_5 &= 11 \\ -x_2 - 6x_3 + 3x_4 - x_5 &= -11 \end{aligned}$$

Be sure to display the reduced form of the augmented matrix. [11 points]

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -1 & 1 & 4 \\ 0 & 1 & 6 & -3 & -1 & 11 \\ 0 & -1 & -6 & 3 & -1 & -11 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -1 & 1 & 4 \\ 0 & 1 & 6 & -3 & -1 & 11 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right)$$

$R_1 - 2R_2 \to R_1$

$(-\frac{1}{2})R_3 \to R_3$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -9 & 5 & 3 & -18 \\ 0 & 1 & 6 & -3 & -1 & 11 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$R_1 - 3R_3 \to R_1$

$$\begin{array}{c} \begin{array}{cccccc} & & s & t & & \end{array} \\ \boxed{\left(\begin{array}{ccccc|c} 1 & 0 & -9 & 5 & 0 & -18 \\ 0 & 1 & 6 & -3 & -1 & 11 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)} \end{array}$$

REDUCED

ANSWER

$x_1 = 9s - 5t - 18$
$x_2 = -6s + 3t + 11$
$x_3 = s$
$x_4 = t$
$x_5 = 0$

$s, t \in \mathbb{R}$ parameters

(b) Find the particular solution to the system in (a) such that $x_3 = 2$ and $x_4 = 1$.

[3 points]

$s = x_3 = 2 \quad t = x_4 = 1$

$x_1 = 18 - 5 - 18 = -5$

$x_2 = -12 + 3 + 11 = 2$

$x_3 = 2$

$x_4 = 1$

$x_5 = 0$

ANSWER

$x_1 = -5$
$x_2 = 2$
$x_3 = 2$
$x_4 = 1$
$x_5 = 0$

4. Each day a small food company has available 150 kg of nuts and 90 kg of raisins to be combined and sold as two different trail-mix snack foods called "Tasty" and "Yummy". The Tasty trail-mix contains 1/2 nuts and 1/2 raisins and sells for \$7 per kg. Yummy trail-mix contains 3/4 nuts and 1/4 raisins and sells for \$9.50 per kg. How many kg of each trail-mix should the company prepare per day to maximize revenue? What is the maximum revenue? You may assume that all Tasty and Yummy trail-mixes prepared daily are sold daily. Show all appropriate details and calculations in your solution.

[11 points]

Let x = amount in kg of Tasty sold daily
y = " " " " Yummy " "

We require that $x \geq 0$, $y \geq 0$, $x, y \in \mathbb{R}$.

Revenue function is $Z = R(x,y) = 7x + 9.5y$

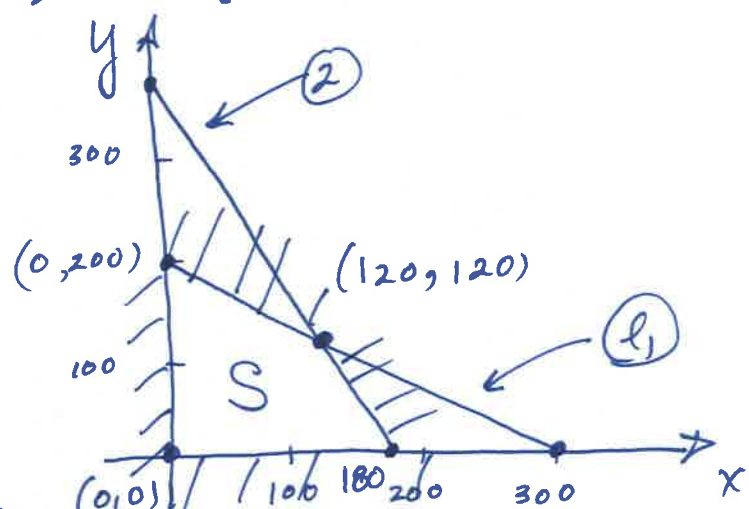
Constraints are $\frac{1}{2}x + \frac{3}{4}y \leq 150$ (nuts)
$\frac{1}{2}x + \frac{1}{4}y \leq 90$ (raisins)

"Integerize" these:
$$\left.\begin{aligned} 2x + 3y &\leq 600 \\ 2x + y &\leq 360 \end{aligned}\right\} (*)$$

LPP is to maximize Z subject to $(*)$; $x, y \geq 0$.

ℓ_1: $2x + 3y = 600$
ℓ_2: $2x + y = 360$

$2y = 240$
$\therefore y = 120$
$x = 120$



S = feasible region, bounded, non-empty, "Non-region" is shaded ///

S has corners $(0,0)$, $(0,200)$, $(120,120)$, $(180,0)$

By FTLP, we can find Max Z by corner-point evaluation

Evaluation:

$Z(0,0) = 0$
$Z(0,200) = 1{,}900$
$Z(120,120) = 1{,}980$
$Z(180,0) = 1{,}260$

$\therefore$ MAX Revenue is \$1,980 when $x = y = 120$.

$$\begin{array}{cc} \begin{array}{ccc} G & S & B \end{array} & \\ \begin{pmatrix} 10 & 5 & 8 \\ 12 & 0 & 4 \\ 0 & 10 & 16 \end{pmatrix} & \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} \end{array}$$

5. In all of this question let $A = (a_{ij}) = \begin{pmatrix} 10 & 5 & 8 \\ 12 & 0 & 4 \\ 0 & 10 & 16 \end{pmatrix}$

The entries in A are interpreted as follows. The columns give the numbers of Gold, Silver, and Bronze medals, respectively, predicted to be won by three countries C_1, C_2, C_3 (represented by the rows) at the 2014 Winter Olympic Games. For example, $a_{23} = 4$ means that C_2 is predicted to win 4 Bronze medals and $a_{31} = 0$ means that C_3 is predicted to win no Gold medals.

(a) State the matrix B so that the entries in $A + B$ represents a revised prediction where: each country wins 3 more Gold medals, each country increases its number of Silver medals won by 20%, and each country decreases its number of Bronze medals won by 25% [4 points]

$$A+B = \begin{pmatrix} 13 & 6 & 6 \\ 15 & 0 & 3 \\ 3 & 12 & 12 \end{pmatrix}$$

$$\therefore B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & 0 & -1 \\ 0 & 2 & -4 \end{pmatrix}$$

(b) Find det(A). [5 points]

$$\det(A) = 10\begin{vmatrix} 0 & 4 \\ 10 & 16 \end{vmatrix} - 5\begin{vmatrix} 12 & 4 \\ 0 & 16 \end{vmatrix} + 8\begin{vmatrix} 12 & 0 \\ 0 & 10 \end{vmatrix}$$

$$= 10(-40) - 5(192) + 8(120)$$

$$= -400 - 960 + 960$$

$$= -400$$

(c) Is there a matrix C so that the entries in AC give the same revised prediction described in part (a)? Justify your answer. If there is such a matrix C, you need not actually find it, you just need to justify why it exists. [4 points]

Consider solving for C where:

$$AC = A + B$$

$$AC - AI = B$$

$$A(C - I) = B$$

$\because \det(A) \neq 0$

$\therefore A^{-1} \exists$

$$\to C - I = A^{-1}B$$

$$\boxed{C = A^{-1}B + I}$$

ANSWER IS "YES"... C IS ABOVE

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix, $n \geq 2$, and let $P = A^T A$.
Prove: if the diagonal entries in P are all equal to 0, then $A = 0$. [6 points]

$$P = A^T \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

COL i

ROW i

$$= \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$P = (P_{ij})$ Assume $P_{ii} = 0$. Let $i \in \{1, \ldots, n\}$.

$$P_{ii} = 0 = \sum_{k=1}^{n} (a_{ki})(a_{ki}) = \sum_{k=1}^{n} a_{ki}^2$$

$\Rightarrow$ $a_{ki} = 0$ all $k = 1, \ldots, n$.

$\therefore$ $a_{ki} = 0$ all $i = 1, \ldots, n$ and all $k = 1, \ldots, n$.

$\Rightarrow$ $A = 0$. ∎

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Basic Statistics (*)

$N = 478$ = # of students who wrote the test

$P = 452$ = # of students who passed test

Mark	# of students	% of class
90's	35	7.3
80's	116	24.3
70's	141	29.5
60's	101	21.1
50's	59	12.3
40's	20	4.2
30's	6	6.3
20's	0	
10's	0	
1's	0	

% ≥ 80% ~ 31.6%

% ≥ 70% ~ 61.1%

$\overline{X}$ = AVERAGE ~ 72.2%

(*) Compiled before any regrading.