University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

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Date: March 1, 2008
Duration: 110 minutes

Provide the following information:

(Print) Surname: Solutions

(Print) Given Name(s): ______

Student Number: ______

Signature: ______

Tutorial Number (e.g. TUT0033): ______

Carefully circle the name of your Teaching Assistant:

Marc CASSAGNOL
Paula EHLERS
Wenbin KONG
Carmen KU
Chris LUI
Hifzur PATEL
Molu SHI
Sean TRIM
Alfred YIP
Yichao ZHANG
Sam (Zhe) ZHOU

Read these instructions:

1. This midterm test has 11 numbered pages. It is your responsibility to ensure that, at the beginning of the test, all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and indicate clearly the location of your continuing work.
3. With the exception of the Multiple Choice questions, full points are awarded only for solutions that are correct, complete, and sufficiently display concepts and methods of MATA33.
4. You may use **one** standard hand-held calculator. The following electronic devices are forbidden: laptop computers, Blackberrys, cell-phones, I-Pods, and MP-3 players.
5. Extra paper, notes, or textbooks are forbidden.

NOTE: The (REMARKS) given @ the end of many questions indicate how some part-marks were allotted. This will help you to understand how your test was graded.

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6
b	b	a	b	d	c

4 points each

Do not write anything in the boxes below.

2 pts for ALL OF COVER COMPLETE

0 pts if NOT ALL COVER COMPLETE

No part marks!

Info.	Part A
2	24

Part B

1	2	3	4	5	6	7
10	7	8	17	10	12	10

Total
100

(Reasoning is supplied, but not req^d)

Part A - Multiple Choice For each of the following, **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 4 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded but a small workspace is provided for your calculations and rough work.

Here's some reasoning (NOT REQ'D)

1. If $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix}$ then $A^{-1} + B^T A =$

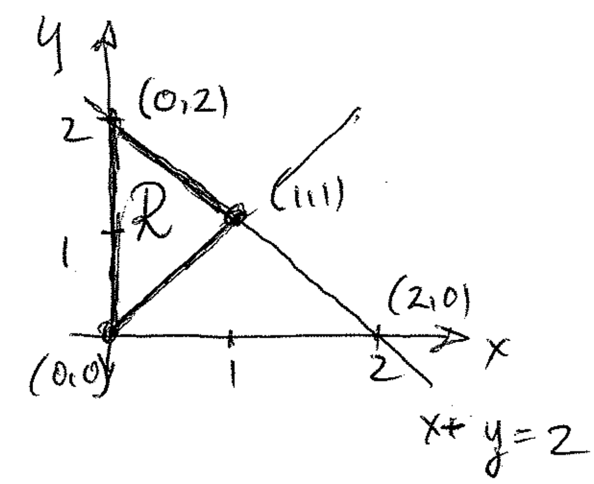
   (a) $\begin{bmatrix} 6 & 17 \\ 29 & 35 \end{bmatrix}$ (b) $\begin{bmatrix} 12 & 9 \\ 25 & 41 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & 8 \\ 8 & 21 \end{bmatrix}$ (d) none of (a), (b), or (c)

$A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ $B^TA = \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 13 \\ 27 & 38 \end{bmatrix}$

$A^{-1} + B^TA = \begin{bmatrix} 12 & 9 \\ 25 & 41 \end{bmatrix}$

2. Assume that when x units of product **X** are sold and y units of product **Y** are sold, the profit is $P = 4x + y$ dollars. If x and y are subject to the constraints $x \leq y$, $x + y \leq 2$, and $x, y \geq 0$ then the maximum profit will occur amongst:

   (a) (0,0), (1,1), (2,0) (b) (0,0), (0,2), (1,1) (c) (0,2), (1,1), (2,0)
   (d) none of (a), (b), or (c)



3. Let $A = [a_{i,j}]$ be an $n \times n$ matrix, $n \geq 2$. The following properties are equivalent to A being invertible:

   (a) The reduced matrix of A is the $n \times n$ identity.
   (b) The product of the diagonal entries in A is not equal to zero (i.e. $a_{1,1}a_{2,2}...a_{n,n} \neq 0$)
   (c) $det(A) > 0$
   (d) All of the above.

$\left|\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right| = 4$, but $A^{-1} \nexists$ $\therefore$ (b) X

$\left|\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}\right| = -2$ but $A^{-1} \exists$ so A^{-1} not equivalent to (c) X

4. Let a and c be real constants and let x and y be variables. The system of linear equations

$$\begin{aligned} cx - 8y &= a \\ 2x + cy &= c \end{aligned}$$

has a unique solution

(a) only if $c \neq \pm 4$

(b) for all values of a and c.

(c) only if $a \neq 0$ and $c \neq 0$

(d) for no values of a and c.

$A = \begin{bmatrix} c & -8 \\ 2 & c \end{bmatrix}$ Unique sol$^{\underline{n}}$ $\Leftrightarrow \det(A) \neq 0$

$\det(A) = c^2 + 16 \neq 0 \quad \forall c, a \in \mathbb{R}$

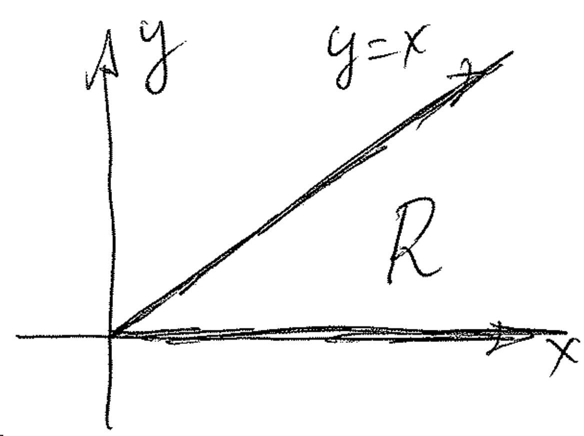
5. Let $Z = 2x - 3y$ and let R represent the feasible region defined by the two inequalities $y \geq 0$ and $y \leq x$. We may conclude that

(a) Z has both a maximum and a minimum in R.

(b) Z has a maximum, but no minimum in R.

(c) Z has a minimum, but no maximum in R.

(d) Z has no optimal solution in R.



(a) $\times$ (b) $\times$ $Z(x, 0) \to \infty$ as $x \to \infty$

(c) $\times$ (a) $\times$ $Z(x, x) = -3x \to -\infty$ as $x \to \infty$

6. A general homogeneous system of $n \geq 2$ linear equations in n unknowns

(a) has only the trivial solution.

(b) has infinitely many solutions.

(c) has at least one solution.

(d) may not have any solutions.

(a) $\times$ $\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \end{aligned}$

(b) $\times$ $\begin{aligned} x + y &= 0 \\ x - y &= 0 \end{aligned}$

(d) $\times$ $x_1, \dots, x_n = 0$ is always a sol$^{\underline{n}}$

(Did you print the Multiple Choice answers in the boxes at the top of page 2 ?)

Part B - Full Solution Problem Solving

1. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

Use the method of row reduction to determine whether A is invertible. If it is, find A^{-1}. If it is not, then briefly justify why this is the case. Show all of your work and correct notation for row operations. [10 points]

$$[A|I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - 4R_1 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \quad R_3 + 3R_2 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\left(\tfrac{1}{2}\right)R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

$$\begin{array}{l} R_1 - 3R_3 \to R_1 \\ R_2 \to 2R_3 \to R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

$\underbrace{\hphantom{xxxxxxx}}_{3\times 3 \text{ Id}}$

$\therefore$ A is invertible (ie A is row reducible to I_3)

$\therefore A^{-1} \exists$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

REMARKS

1. If no row operations are indicated, deduct 1 pt.
2. If there is no mentioning of A being row reduced to I, deduct 1 pt.
3. Gave liberal partmarks for "clear progress".

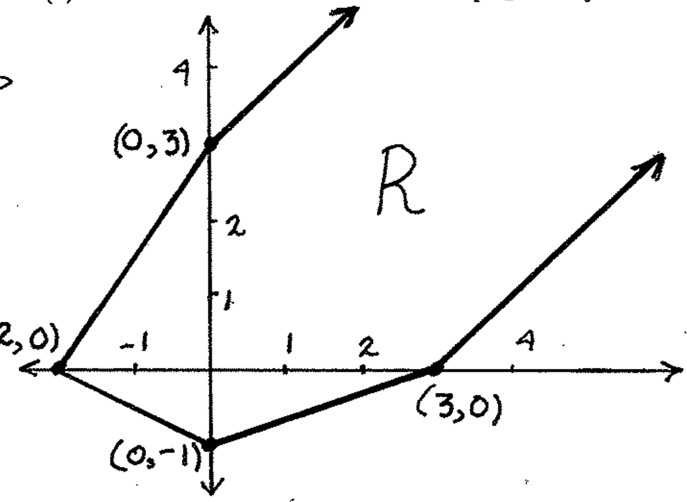
Remark Deduct 1 pt if there is no mention of any theory.

2. Let $r > 0$ be a constant and let $Z = -3rx + 2ry$ where (x, y) lies in the feasible region R shown at the right (the boundaries of R are included in R). Assuming that Z has a maximum value on R, find this maximum value and all point(s) where this value occurs. [7 points]

$\because$ Z has a max $\Rightarrow$ it occurs @ a corner point.

$Z(0,3) = 6r$ $\quad Z(0,-1) = -2r$

$Z(-2,0) = 6r$ $\quad Z(3,0) = -9r$



Multiple Optimal solⁿ theory

$\Rightarrow$ Max Value is $6r$ and occurs at every point on the segment joining $(-2,0)$ and $(0,3)$

3. Determine the value(s) of the real number s for which the system [8 points]

$$\begin{aligned} sx - 2sy &= -1 \\ 3x + 6sy &= 4 \end{aligned}$$

has a unique solution. For these value(s) of s, use Cramer's rule to solve for x and y.

Let $A = \begin{bmatrix} s & -2s \\ 3 & 6s \end{bmatrix}$ Unique solⁿ $\Leftrightarrow$ A^{-1} exists $\Leftrightarrow$ $\det(A) \neq 0$

$\det(A) = 6s^2 + 6s = 6s(s+1) \Rightarrow \underbrace{\det(A) \neq 0 \ \forall s \in \mathbb{R},\ s \neq 0, -1}_{\text{Unique sol}^n \text{ for these } s \text{ values}}$

Cramer's rule: for $s \neq 0, -1$

$$x = \frac{\det\begin{bmatrix} -1 & -2s \\ 4 & 6s \end{bmatrix}}{\det(A)} = \frac{-6s + 8s}{6s(s+1)} = \boxed{\frac{1}{3(s+1)} = x}$$

$$y = \frac{\det\begin{bmatrix} s & -1 \\ 3 & 4 \end{bmatrix}}{\det(A)} = \boxed{\frac{4s+3}{6s(s+1)}} \text{ "} y \text{"}$$

REMARKS

① 2 pts for unique $\Leftrightarrow s \neq 0, -1$

② 6 pts for getting x & y

4. Minimize $Z = 2x + y$ subject to the constraints: $x + 2y \geq 5$, $x \leq 3y$, $4x + 5y \leq 34$, $x \geq 1$, and $y \leq 6$: Your solution should clearly show the feasible region, labeled corner points, the minimum value of Z, and all point(s) where the minimum occurs. [17 points]

L_1 is $x + 2y = 5$ Intercepts $(5,0)$, $(0, 5/2)$

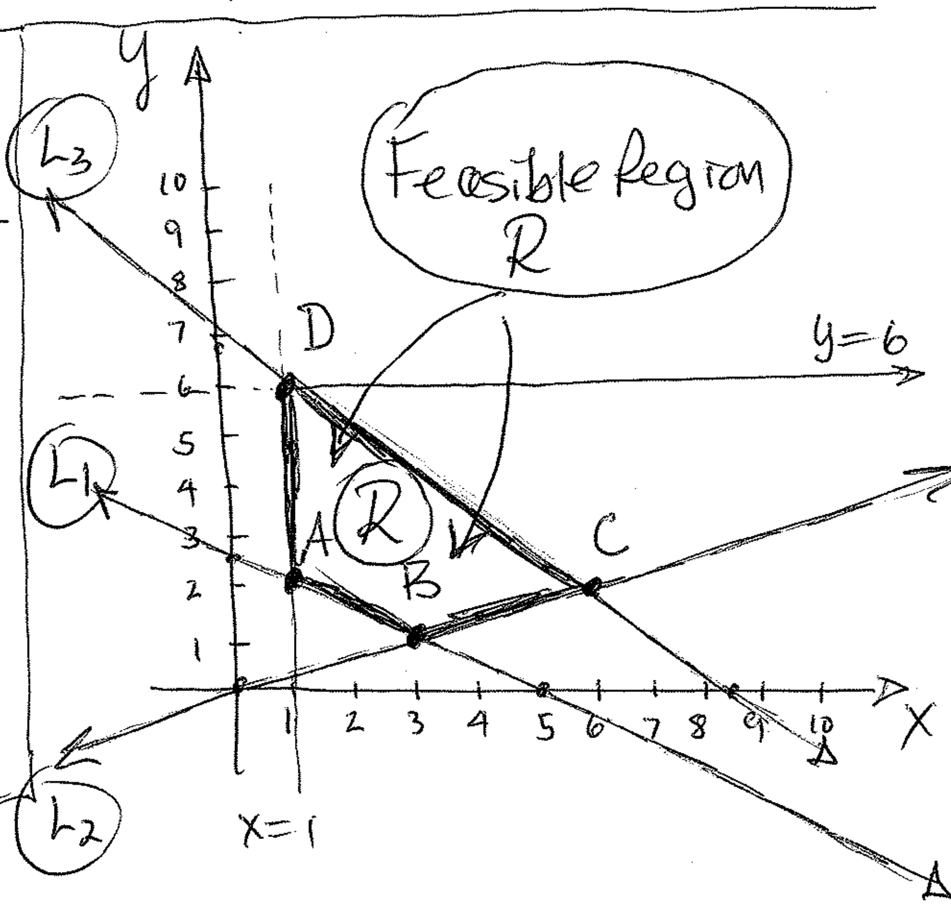
L_2 is $x + 3y$ Intercept $(0,0)$, Extra pt. $(3,1)$

L_3 is $4x + 5y = 34$ Intercepts $(\frac{17}{2}, 0)$, $(0, \frac{34}{5})$

L_1 crosses $x=1$ line @ $(1,2)$

L_2 crosses $x=1$ line @ $(1, 1/3)$ & crosses L_1 @ $(3,1)$

L_3 crosses $x=1$ line @ $(1,6)$ and crosses L_2 @ $(6,2)$



Corner points for the feasible region R:

$A = (1, 2)$ $B = (3, 1)$ $C = (6, 2)$ $D = (1, 6)$

R is $\neq \phi$ & bounded, and Z is linear

FTLP $\Rightarrow$ Z is minimized at a corner point.

$Z(A) = 4$ $Z(B) = 7$ $Z(C) = 14$ $Z(D) = 8$

$\boxed{\text{Minimum Value of } Z \text{ is } 4 \text{ occuring at } A = (1,2)}$

(FTLP = Fundamental Theorem of Linear Programming)

REMARKS ① 2 points for citing $\neq \phi$, bded & corner pt via FTLP | ② 3 points for all features on diagram

5. Use matrix reduction to solve the system of linear equations [10 points]

$$\begin{aligned} 3x_1 - x_2 + x_3 + x_4 + 14x_5 &= 0 \\ -x_1 + x_2 + 3x_3 + 0x_4 - 6x_5 &= 2 \\ x_1 + 0x_2 + 2x_3 + 0x_4 + 4x_5 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 3 & -1 & 1 & 1 & 14 & 0 \\ -1 & 1 & 3 & 0 & -6 & 2 \\ 1 & 0 & 2 & 0 & 4 & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 4 & -1 \\ -1 & 1 & 3 & 0 & -6 & 2 \\ 3 & -1 & 1 & 1 & 14 & 0 \end{array}\right]$$

$R_2 + R_1 \to R_2$

$R_3 - 3R_1 \to R_3$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 4 & -1 \\ 0 & 1 & 5 & 0 & -2 & 1 \\ 0 & -1 & -5 & 1 & 2 & 3 \end{array}\right]$$

$R_3 + R_2 \to R_3$
$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 4 & -1 \\ 0 & 1 & 5 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 4 \end{array}\right] \end{array}$$

Solution is :

$x_1 = -1 - 2s - 4t$

$x_2 = 1 - 5s + 2t$

$x_3 = s$

$x_4 = 4$

$x_5 = t$

s, t are parameters and can assume any real values

(REMARKS)

① Gave liberal part marks for a solution showing clear knowledge / progress, even if final answer is incorrect.

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$$\overbrace{}^{A}$$

6. (a) Use cofactor expansion along any row or column to find $det\begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

[6 points]

Cofactor expansion along row 3

$$\det(A) = (-1)\begin{vmatrix} -4 & 2 \\ 8 & -9 \end{vmatrix} - 7\begin{vmatrix} 1 & 2 \\ -2 & -9 \end{vmatrix}$$

$$= -1(36-16) - 7(-9+4)$$

$$= -20 + 35 = \boxed{15}$$

REMARKS

① Gave liberal part marks for a cofactor expansion with only arithmetic errors!

② Give at most 3 out of 6 for "up/down product" solution — we want cofactors only

(b) Let A and B be 4×4 matrices such that $det(A) = -2$ and $det(B) = 5$
Find the following:

[6 points]

[2pts] (i) $det(3A^2B)$ $= 3^4(\det(A))^2 \det(B)$

$$= 81(-2)^2(5) = \boxed{1620}$$

[2pts] (ii) $det((\frac{1}{2}B)^{-1})$ $= \det(2(B^{-1})) = 2^4 \dfrac{1}{\det(B)}$

$$= \boxed{16/5}$$

[2pts] (iii) $det(det(A)B^T)$ $= (\det(A))^4 \det(B)$

$$= (16)(5) = \boxed{80}$$

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REMARKS

Deduct ≤ 1 pt per part if only an answer is given with no steps.

(REMARKS)

① For (a), $1\frac{1}{2}$ to calculate + $1\frac{1}{2}$ to interpret

② For (b), liberal part marks for decent progress

7. A small town has three service stations: S_1, S_2, and S_3. Each station sells the same three blends of gasoline: G_1 (regular), G_2 (extra), and G_3 (super). Let $A = [a_{i,j}] = \begin{bmatrix} 98.3 & 102.1 & 108.5 \\ 99.2 & 103.5 & 109.0 \\ 96.8 & 101.2 & 106.9 \end{bmatrix}$ where $a_{i,j}$ = the price in cents that station S_i charges for a litre of gasoline blend G_j

(a) Calculate and iterpret (i.e. use mathematical concepts to briefly explain within the context of the scenario described above) the products MA and AM^T where $M = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$.

[6 points]

$$MA = \begin{bmatrix} \frac{98.3+99.2+96.8}{3} & \frac{102.1+103.5+101.2}{3} & \frac{108.5\ \ 109.0\ \ 106.9}{3} \end{bmatrix}$$

$$= [98.1 \quad 102.3 \quad 108.1]$$

Interpretation: j^{th} entry gives average price in ¢ of blend G_j over all 3 stations; $j = 1,2,3$

$$AM^T = \begin{bmatrix} \frac{98.3+102.1+108.5}{3} \\ \frac{99.2+103.5+109.0}{3} \\ \frac{96.8+101.2+106.9}{3} \end{bmatrix} = \begin{bmatrix} 103.0 \\ 103.9 \\ 101.6 \end{bmatrix}$$

Interpretation: i^{th} entry gives average price in ¢ of gas at station S_i over all 3 blends; $i = 1,2,3$.

(b) Suppose that during a certain 30-minute period, station S_i sells $d_{i,j}$ litres of gasoline G_j Let D be the 3×3 matrix whose i,j entry is $d_{i,j}$ If R_i represents the total gasoline revenue of station S_i in dollars during this 30-minute period, describe using matrix notation how one can calculate R_i using the matrices A and D.

[4 points]

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

Answer

Calculate

$$\boxed{R_i = \frac{1}{100}\left[i^{th} \text{ row of } A\right] \cdot \begin{bmatrix} i^{th} \\ \text{column} \\ \text{of} \\ D^T \end{bmatrix} \quad i = 1,2,3}$$

Example $R_1 = \frac{1}{100}\begin{bmatrix} 98.3 & 102.1 & 108.5 \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \end{bmatrix}$

(NOT REQ D ... For Illustration purposes only)

$$= \left[\frac{98.3 d_{11} + 102.1 d_{12} + 108.5 d_{13}}{100}\right]$$

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