University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

MATA33 Midterm Test Summer 2015

Examiner: A.Pavlov

Date: June 12, 2015
Time: 7:00 pm
Duration: 110 minutes

(Print CAPITALS) LAST NAME: ______________________

(Print) Given Name(s): ______________________

Student Number: ______________________

Signature: Solutions and Basic Stats

Circle your Tutorial Number: T0001 T0002 T0003 T0004 T0005

Read these instructions:

1. This test has 13 pages. It is your responsibility to check at the beginning of the test that all of these 13 pages are included.
2. If you need extra answer space for any question, use the back of a page of the last page. Clearly indicate the location of your continuing work.
3. The following are forbidden at your workspace: calculators, any other kind of electronic aid or device (e.g. cell/smart phones, ipads, iphones, etc.), scrap paper, food, textbooks, bags, pencil/pen carrying cases, drinks in a paper cup or box or similar container that has a removable label.
4. Cell/smart/iphones must be turned off and left at the front of the test room.
5. You are encouraged to write in pen or other ink, not pencil. If any part of your test is written in pencil, then you will be denied any re-grading opportunity.

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6	7
D	G	D	D	E	C	D

Do not write anything in the boxes below.

Info	**Part A**
2	21

Part B

1	2	3	4	5
24	9	13	17	16

Total
100

Part A - Multiple Choice. For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2**. Each right answer earns 3 points and no answer/wrong answer/ambiguous answers earn 0 points. A small workspace is provided for your calculations and rough work.

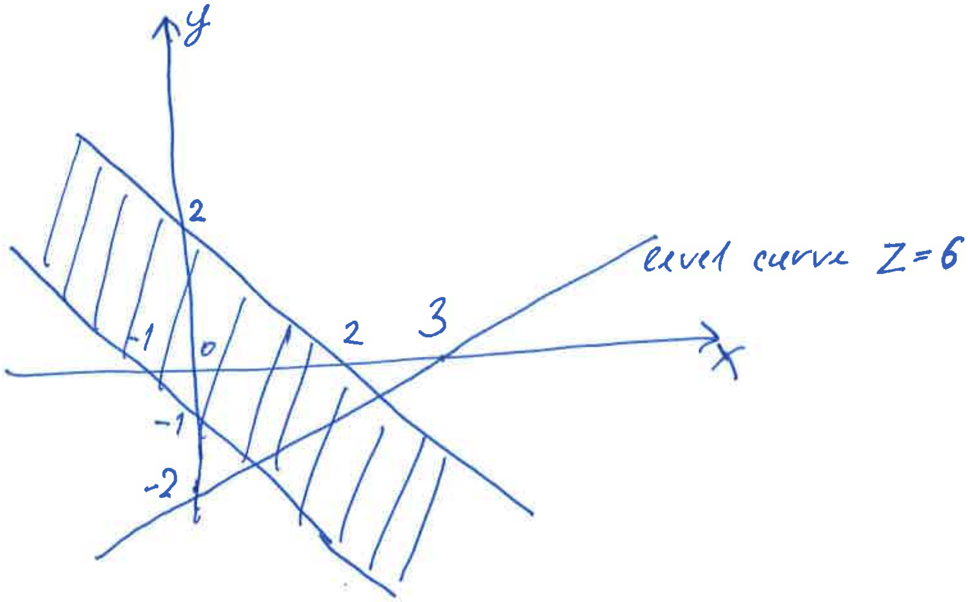
1. A linear programming problem has objective function $Z = 2x - 3y$ subject to constraints $x \leq 2 - y$ and $y \geq -1 - x$. This problem has

✗ (A) A maximum value on the feasible region, but no minimum value.

✗ (B) A minimum value on the feasible region, but no maximum value.

✗ (C) Both maximum and minimum values on the feasible region.

✓ (D) Neither a maximum nor a minimum on the feasible region.



2. Let $A = [A_{ij}]$ be a square matrix of order 4 defined as $A_{ij} = \frac{1}{i+j+1}$. The sum $\sum_{i=1}^{4} A_{ii}$ is

$(A)\,0$ $(B)\,1$ $(C)\,24$ $(D)\,\frac{1}{24}$ $(E)\,-24$ $(F)\,10$ (G) None of (A)-(F).

$$\sum_{i=1}^{4} A_{ii} = \sum_{i=1}^{4} \frac{1}{2i+1} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} = \frac{248}{315}$$

3. Let A be a square matrix of order n, which of the following matrices are symmetric?

$(A)\, A^{\top}A$ $\quad (B)\, A + A^{\top}$ $\quad (C)\, A^2 + A + I$ $\quad (D)$ (A) and (B) $\quad (E)$ (A)-(C)

(F) (B) and (C) $\quad (G)$ None of (A)-(F).

$(A^TA)^T = A^T(A^T)^T = A^TA$ Use $(A^T)^T = A$.

$(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

In general $(A^2+A+I)^T \neq A^2+A+I$.

4. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 6 & 8 \end{bmatrix}$ $C = BA^{-1} + A$. What is $c_{11}c_{22}$?

$(A)\, 0$ $\quad (B)\, 1$ $\quad (C) - 30$ $\quad (D) - 12$ $\quad (E)\, 12$ $\quad (F)\, 4$ $\quad (G)$ None of (A)-(F).

$$A^{-1} = \frac{1}{2\cdot 2 - 3\cdot 1}\begin{bmatrix} +2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$BA^{-1} = \begin{bmatrix} 0 & 1 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -12 & 10 \end{bmatrix}$$

$$BA^{-1} + A = \begin{bmatrix} -3 & 2 \\ -12 & 10 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -9 & 12 \end{bmatrix}$$

$c_{11} = -1$

$c_{22} = 12$

$c_{11}c_{22} = -12$

5. Let A and B be as in the previous question. What is $det(2A^{11}B^2)$?

$(A)\,0$ $(B)\,2$ $(C)-36$ $(D)-144$ $(E)\,144$ $(F)\,36$ (G) None of (A)-(F).

$det(A)=1, \quad det(B)=-6$

$det(2A^{11}B^2)=2^2 det(A)^{11} det(B)^2=4\cdot 1^{11}\cdot(-6)^2=144$

6. How many of the following statements are correct?

X (a) Matrix A is invertible if and only if $det(A)=0$.

V (b) Homogeneous system has infinitely many solutions if it has fewer equations than variables.

X (c) Product of two symmetric matrices is a symmetric matrix.

V (d) If a matrix A is invertible then A^2 is invertible.

$(A)\,0$ $(B)\,1$ $(C)\,2$ $(D)\,3$ $(E)\,4$.

7. Let $A=\begin{bmatrix}2 & 2\\ 1 & 3\end{bmatrix}$. For which values of t the matrix $A-tI$ is not invertible?

$(A)\,t=1$ $(B)\,t=4$ (C) any t $(D)\,(A)$ and (B) $(E)\,t=-1$ $(F)\,t=-4$
(G) None of (A)-(F).

$det(A-tI)=0$

$\begin{vmatrix}2-t & 2\\ 1 & 3-t\end{vmatrix}=0$

$(2-t)(3-t)-2=0$

$6-3t-2t+t^2-2=0$

$t^2-5t+4=0$

roots: $t_1=1,\ t_2=4$.

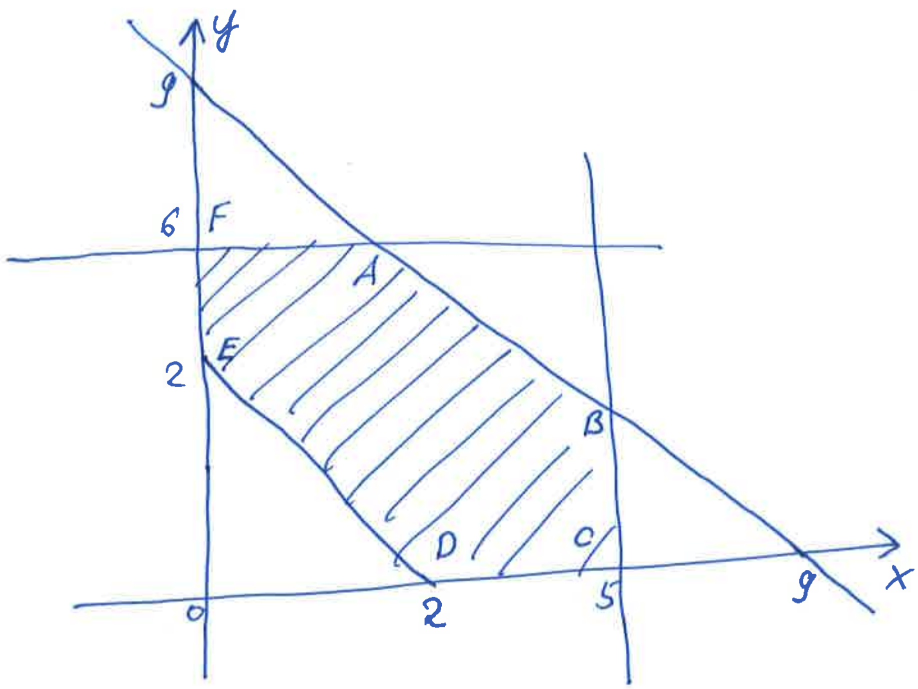
***** DID YOU PUT THE ANSWERS IN THE BOXES ON PAGE 2? *****

Part B - Full Solution Problem Solving. Full points will be awarded for your solutions if and only if they are correct, complete, and show sufficient relevant concepts from MATA33.

1. (a) Find the maximum and minimum values of the linear function $Z = 5x - 7y$ subject to constraints

$$x + y \leq 9,$$
$$x + y \geq 2,$$
$$x \leq 5,$$
$$y \leq 6,$$
$$x \geq 0, y \geq 0.$$

Your answer should include a neat, labeled feasible region, the location of all corner points, and sufficient computation details. (15 points)



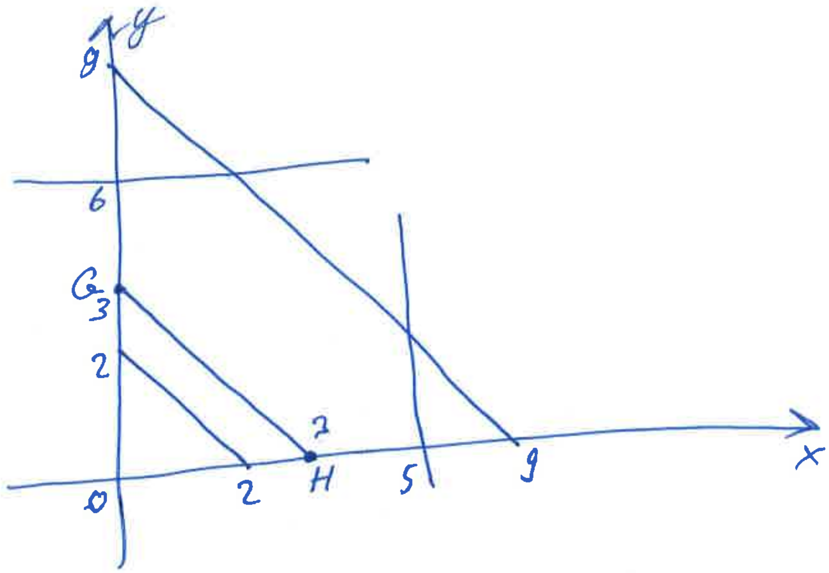
$A = (3, 6)$
$B = (5, 4)$
$C = (5, 0)$
$D = (2, 0)$
$E = (0, 2)$
$F = (0, 6)$

$Z(A) = -27$
$Z(B) = -3$
$Z(C) = 25$
$Z(D) = 10$
$Z(E) = -14$
$Z(F) = -42$

The min. value is -42 at the point F, the max. value is 25 at the point C.

(b) Find the maximum and minimum of the same function satisfying the same constraints plus extra constraint $x + y = 3$. (9 points)

Intersection of the feasible region of part (a) and line $x+y=3$ is the line segment joining points G and H:



Corner points are G and H.

$Z(G) = -21$

$Z(H) = 15$

The min. value is -21 at the point G, the max. value is 15 at the point H.

3. (a) Find A^{-1} by the method of row reduction if

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}.$$

(9 points)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2-2R_1]{R_2+3R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_2]{R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 7 & 2 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+3R_3]{R_1+5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -18 & -3 & 5 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix}$$

(b) Use your result in (a) to solve linear equation $AX = B$, where $B = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$. (4 points)

$$X = A^{-1}B = \begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -33 \\ -21 \\ -8 \end{bmatrix}$$

4. (a) Compute $AA^{\top}B^{-1} - 2I$ if $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. (9 points)

$$AA^T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{det(B)} adj(B) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$AA^TB^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 7 \end{bmatrix}$$

$$AA^TB^{-1} - 2I = \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix}$$

(b) Compute $B^3 + A$, where matrices A and B are the same as in part (a). (2 points)

B^3 is 2×2 matrix, A is 2×3 matrix, the sum is not defined.

(c) Compute $det(A^{\top}A)$. (6 points)

$$A^TA = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$det(A^TA) = 0.$

5. (a) Use the method of reduction to solve the system

$$\begin{aligned} x - y - 3z &= 0 \\ x + y - z &= 0 \\ 2x - y - 5z &= 0 \end{aligned}$$

(11 points)

$$\begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & -1 \\ 2 & -1 & -5 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 1 & -1 & -3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-R_2}$$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Leading entries are circled. Column containing the leading entries correspond to basic variables: x, y.

Free variables: z.

Parameter: $z = r$

1st row: $x - 2z = 0 \iff x = 2r$

2nd row: $y + z = 0 \iff y = -r$

The solution

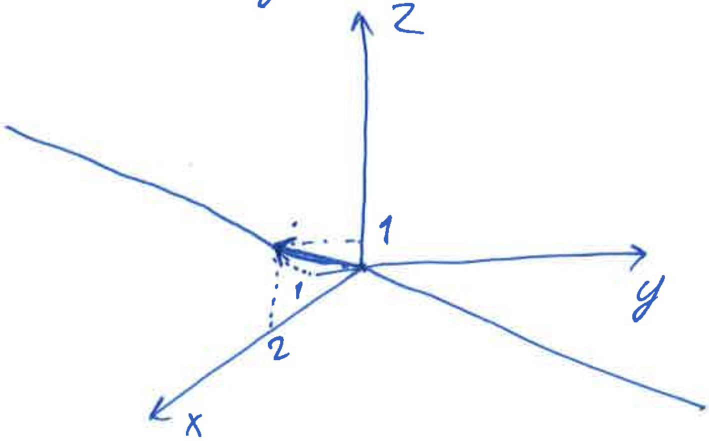
$x = 2r$

$y = -r$

$z = r$

(b) Give a geometric interpretation of the solution of the system. (2 points)

This is a line in the 3d space, passing through the origin, direction vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$



(c) Find all solutions of the system $AX = B$, where A is the coefficient matrix from part (a), if $B = \begin{bmatrix} 2 \\ 1 \\ \frac{7}{2} \end{bmatrix}$, and column vector $\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$ is known to be a solution of the non-homogeneous system $AX = B$. (3 points)

Do not repeat reduction for the non-homogenous system.

Let $C = \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \end{bmatrix}$

$AC = B$

Denote the solution of the non-homogeneous system Z: $AZ = B$ and the solution of the homogeneous system X: $AX = 0$

$AZ - AC = B - B = 0$

$A(Z - C) = 0$

$Z - C = X$

$Z = C + X$

$Z = \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

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Basic statistics

# of students who wrote the test: 148

test average: 69

- About 93.2% of 148 students got a score of $\geq 50\%$
- About 79.7% of 148 students got a score of $\geq 60\%$
- About 58.1% of 148 students got a score of $\geq 70\%$

90's — 3.4%
80's — 19.6%
70's — 35.1%
60's — 21.6%
50's — 13.5%
40's — 1.4%
30's — 4.1%
20's — 0%
10's — 0.7%