University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test # 1

MATA33 - Calculus for Management II

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Duration: 110 minutes

Provide all of the following information:

(Print) Surname: SOLUTIONS

(Print) Given Name(s): ______________________

Student Number: ______________________

Signature: ______________________

Tutorial Number (e.g. TUT0033): ______________________

Carefully circle the name of your Teaching Assistant:

Shuyuan (Catherine) BAO	Carmen KU	Suj SRISKANDARAJAH
Jaehyun CHO	Paul LI	Xinling (Elena) WANG
Duy Minh DANG	Alex LUCAS	Chenchen (Audrey) WU
Wenbin KONG	Chris LUI	Alfred YIP

Carefully read these instructions:

1. This test has 10 numbered pages. It is your responsibility to ensure that all of these pages are included.
2. In Part A, enter your letter choice in the boxes at the top of page 2. **Marker(s) will only look in those boxes for your answers to Part A questions.**
3. In Part B, put your solutions in the work space provided. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
4. You may use **one** standard hand-held calculator (graphing is acceptable). The following electronic devices are forbidden at your workspace: laptop computer, Blackberry, cell-phone, I-Pod, MP-3 player, or other similar electronic storage/retrieval devices.
5. Extra paper, notes (either visibly or in a pencil/carrying case), and textbooks are forbidden at your workspace.
6. Tests written in pencil will be denied any re-marking privilege. It is strongly recommended that you write in pen or other ink.

Test 1 Solutions

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5
c	c	b	a	d

Do not write anything in the boxes below.

Info	Part A
3	20

Part B

1	2	3	4	5	6
20	12	11	11	11	12

Total
100

Part A - Multiple Choice Questions Print the letter of the answer you think is correct in the mark box at the top of page 2. Each right answer earns 4 points. Each blank mark box or wrong answer earns 0 points. A small space is provided for your rough work.

1. If $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & -2 \end{bmatrix}$ and $AB + B = -2C$ then C equals

   (a) $\begin{bmatrix} -1 & 7 \\ -1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ -4 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -6 \\ 1 & -9 \end{bmatrix}$ (e) none of (a) - (d)

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ -4 & 16 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 14 \\ -2 & 14 \end{bmatrix}$$

$$\therefore \ C = \begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix}$$

2. A given system of m homogeneous linear equations in n variables where $n > m \geq 1$

   (a) has only the trivial solution (b) has a unique non-trivial solution
   (c) has infinitely many solutions (d) may not have any solutions
   (e) can exist such that none of (a) - (d) are true

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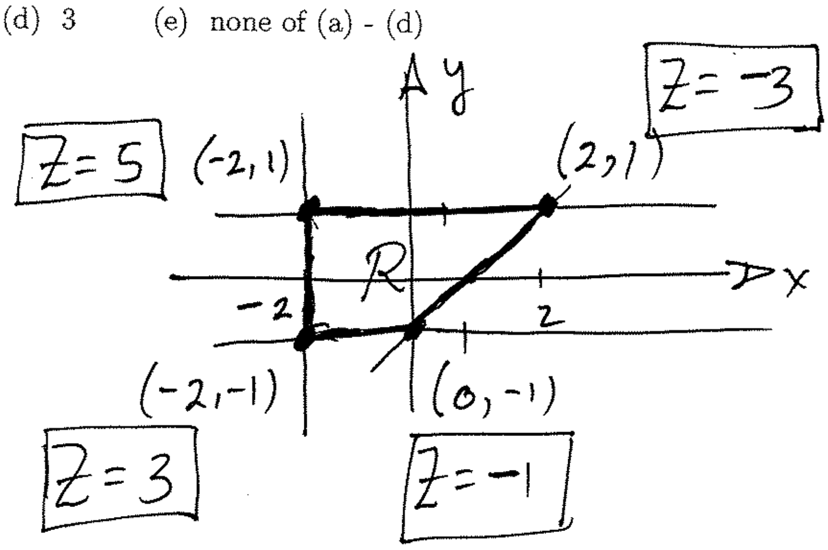
3. The maximum value of $Z = -2x + y$ subject to $x \geq -2$, $x - y \leq 1$ and $-1 \leq y \leq 1$ is

   (a) 6 (b) 5 (c) 4 (d) 3 (e) none of (a) - (d)



4. Let $A = [a_{ij}]$ be an 8×8 matrix where $a_{ij} = i^2 - 2i + j$. Exactly how many entries in A are equal to 4 ?

(a) 3 (b) 2 (c) 1 (d) more than 3 (e) none of (a) - (d)

$a_{ij} = i(i-2)+j$

i	j	
1	5	$\to 4$ ✓
2	4	$\to 4$ ✓
3	1	$\to 4$
4	x	

If $i \geq 4 \to a_{ij} > 8$ x

5. Exactly how many of the following statements are always true ?

(i) Equivalent matrices have the same solution. x

(ii) If the product of two matrices equals the zero matrix, then at least one of the two matrices must be the zero matrix. x

(iii) Different matrices of the same size can have the same reduced form. ✓

(iv) If C and D are matrices and the product CD is defined, then so is the product DC. x

(a) 4 (b) 3 (c) 2 (d) 1 (e) 0

(i) False because the concept "solution" does not apply to Equivalent matrices. We do have that equivalent systems of eq^{ns} have the same solutions.

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Neither of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is O matrix

(Are your Multiple Choice answers in the mark boxes at the top of page 2 ?)

(iv) C $m \times n$

D $n \times p$ CD is defined
DC is not defined if $m \neq p$

Only (iii) is always true.

Part B - Full Solution Problem Solving Full points are awarded for solutions that are numerically correct and sufficiently display concepts and methods in the curriculum of MATA33.

1. Find the maximum and minimum values of the function $Z = -3x + 6y$ subject to the constraints $x - y \geq -3$, $x + y \leq 7$, $x - 2y \leq 4$ and $x, y \geq 0$.

   (To earn full points, your solution must include a neat, labeled diagram of the feasible region, the location of all points where Z is optimized, and all of your calculations.) [20 points]

<u>Solve</u> $y = x + 3$

$y = -x + 7$

$2y = 10 \quad y = 5$

$C(2,5)$

<u>Solve</u> $y = -x + 7$

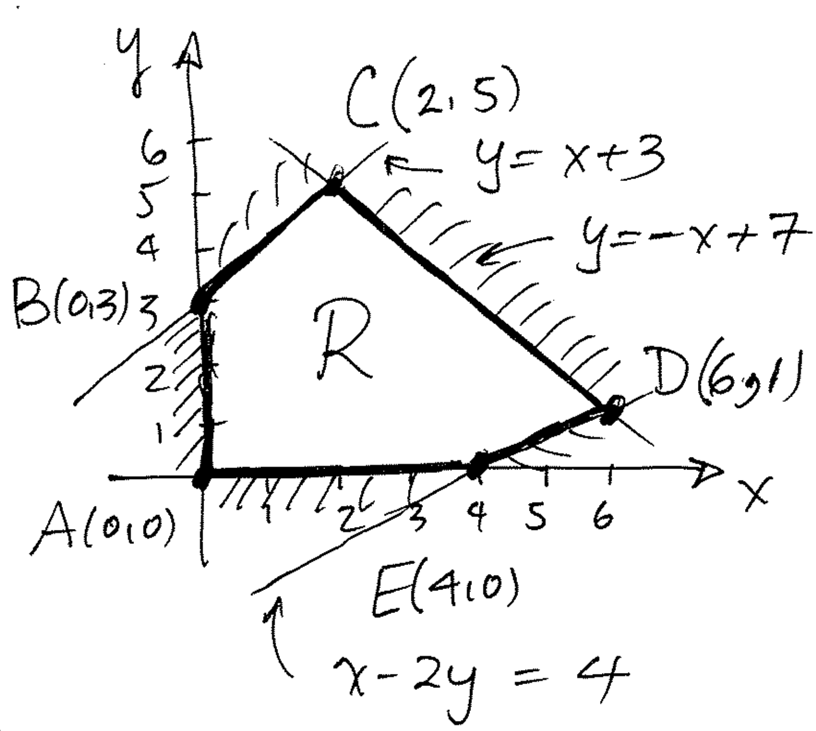
$x - 2y = 4$

$x - 2(-x+7) = 4$

$3x = 18 \quad x = 6$

$y = 1$

$D(6,1)$



Points (ie: corner points) are:

$A(0,0)$ $C(2,5)$ $E(4,0)$

$B(0,3)$ $D(6,1)$

Feasible region is in & on boundaries for R as drawn — R is $\neq \emptyset$, bded.

By FTLP, evaluate @ corners!

$Z(A) = 0 \quad Z(B) = 18 \quad Z(C) = 24$

$Z(D) = -12 \quad Z(E) = -12$

MAX $Z = 24$ @ C

MIN $Z = -12$ @ all pts on segment $\overline{DE}$.

2. Use the method of reduction to solve the system of linear equations [12 points]

$$\begin{aligned} 2x + 5y + 3z &= 3 \\ x + 2y + 3z &= 5 \\ x \qquad + 8z &= 17 \end{aligned}$$

(To obtain full points, your answer must include the reduced form of the augmented matrix)

Augmented matrix

$$A = \left[\begin{array}{ccc|c} 2 & 5 & 3 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 0 & 8 & 17 \end{array}\right]$$

Reduce.

$R_1 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 1 & 2 & 3 & 5 \\ 2 & 5 & 3 & 3 \end{array}\right]$

$R_2 - R_1 \to R_2$
$R_3 - 2R_1 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 2 & -5 & -12 \\ 0 & 5 & -13 & -31 \end{array}\right]$

$\frac{1}{2} R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & -5/2 & -6 \\ 0 & 5 & -13 & -31 \end{array}\right]$

$-13 + \frac{25}{2} = -\frac{1}{2}$

$R_3 - 5R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & -5/2 & -6 \\ 0 & 0 & -\frac{1}{2} & -1 \end{array}\right]$

$-2R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & -5/2 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$R_2 + \frac{5}{2} R_3 \to R_2$
$R_1 - 8R_3 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ ← Reduced form

$\boxed{\text{Solution} \quad x = 1 \quad y = -1 \quad z = 2}$

3. If $B = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$, find all value(s) of a, b and c such that $BB^T = \begin{bmatrix} a-b+20 & c^2-42 \\ c^2-42 & a+b-31 \end{bmatrix}$

[11 points]

$$BB^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix} = \begin{bmatrix} a-b+20 & c^2-42 \\ c^2-42 & a+b-31 \end{bmatrix}$$

$\therefore c^2 - 42 = -17$

$c^2 = 25$

$c = \pm 5$

$a - b + 20 = 21$

$a - b = 1$

$a + b - 31 = 34$

$a + b = 65$

$2a = 66 \rightarrow \boxed{a = 33 \quad b = 32 \quad c = \pm 5}$

4. Let $A = \left[\begin{array}{cccc|c} -2 & 2 & 0 & -10 & -12 \\ 2 & -2 & 3 & 7 & 24 \\ 0 & 0 & 3 & -3 & 12 \end{array}\right]$ State the system of linear equations whose augmented matrix is A. Find the reduced form of A and hence solve the system.

[11 points]

A = Augmented matrix $\Rightarrow$ put in line.

System is
$$\begin{aligned} -2x + 2y - 10w &= -12 \\ 2x - 2y + 3z + 7w &= 24 \\ 0 \qquad 3z - 3w &= 12 \end{aligned}$$

$$A \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 5 & 6 \\ 2 & -2 & 3 & 7 & 24 \\ 0 & 0 & 3 & -3 & 12 \end{array}\right]$$

$$R_2 - 2R_1 \left[\begin{array}{cccc|c} 1 & -1 & 0 & 5 & 6 \\ 0 & 0 & 3 & -3 & 12 \\ 0 & 0 & 3 & -3 & 12 \end{array}\right]$$

$$R_3 - R_2 \left[\begin{array}{cccc|c} 1 & -1 & 0 & 5 & 6 \\ 0 & 0 & 3 & -3 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\tfrac{1}{3}R_2 \rightarrow R_2 \quad \begin{array}{c} \begin{array}{ccccc} x & y & z & w & \end{array} \\ \left[\begin{array}{cccc|c} 1 & -1 & 0 & 5 & 6 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x, z basic

y, w free

Sol^{ns}:

$x = 6 + r - 5s$

$y = r$

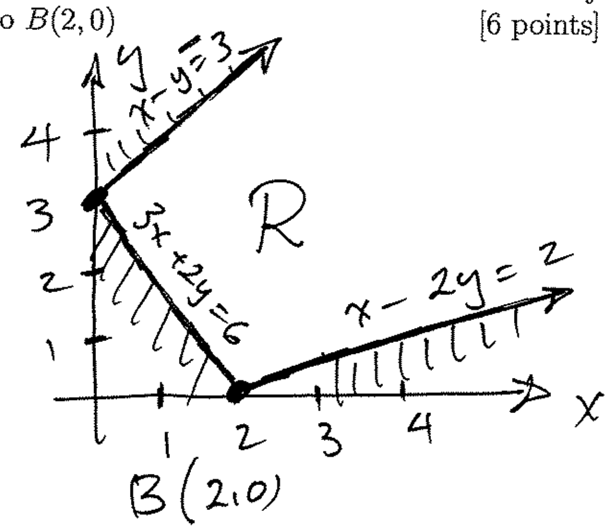
$z = 4 + s$

$w = s$

$r, s \in \mathbb{R}$ are parameters

5. Consider an objective function of the form $Z = ax + by$ where a and b are positive constants. Let R represent the feasible region for the constraints $3x + 2y \geq 6$, $x - y \geq -3$ and $x - 2y \leq 2$ (R is illustrated below. The boundaries of R are included in R).

(a) Find the value of the constants a and b such that Z has a minimum value of 25 at every point on the line segment joining $A(0,3)$ to $B(2,0)$ [6 points]



Question states that

Z has a min

@ this occurs

on segment AB

$Z(0,3) = 3b = 25$

$Z(2,0) = 2a = 25$

$\therefore \; a = \frac{25}{2} \quad b = \frac{25}{3}$

(b) Explain why no choice of positive constants a and b will yield a maximum value for Z on R. [5 points]

Assume $a, b > 0$ are arbitrary.

The line (ie ray) $y = x+3$, $x \geq 0$

is a boundary of R. Evaluation gives

$Z(x, x+3) = ax + b(x+3)$

$= (a+b)x + 3b \longrightarrow \infty$

as $x \to \infty$ on the boundary $y = x+3$.

$\therefore$ Z has no max

on R for any value of $a, b > 0$

m n

6. Four MBA students take the same six courses at graduate business school. Their final marks in these courses are recorded (out of 100) in the matrix $M = [m_{ij}]$ where m_{ij} = the final mark in course j for MBA student i.

M is 4×6

   (a) State the matrices C and Q such that (i) the entries in the product MQ^T are the overall average final marks for each student and (ii) the entries in the product CM are the overall average final marks for each course. [3 + 3 points]

$$Q = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad Q^T \text{ is } 6\times 1 \checkmark$$

$$C = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \qquad C \text{ is } 1\times 4 \checkmark$$

MQ^T is 4×1 ✓ CM is 1×6

   (b) State the matrix R such that the entries in the product RM represent a 5% increase in the final grades for all students in all courses. [3 points]

$$R = 1.05\, I_4 = \begin{bmatrix} 1.05 & 0 & 0 & 0 \\ 0 & 1.05 & 0 & 0 \\ 0 & 0 & 1.05 & 0 \\ 0 & 0 & 0 & 1.05 \end{bmatrix}$$

RM is 4×6

$I_4 = 4\times 4$ identity.

   (c) Suppose the marks in M are altered as follows: the final mark for each student in Course #2 is reduced by 3 points, the final mark for each student in Course #5 is increased by 5 points, and all other marks remain unchanged. State the matrix K so that the sum $K + M$ has entries that reflect the alteration of the marks as described above. [3 points]

$$K = \begin{bmatrix} 0 & -3 & 0 & 0 & 5 & 0 \\ 0 & -3 & 0 & 0 & 5 & 0 \\ 0 & -3 & 0 & 0 & 5 & 0 \\ 0 & -3 & 0 & 0 & 5 & 0 \end{bmatrix}$$

(K must be 4×6 so that the sum $K+M$ is defined)