Solutions & Basic Statistics

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiners: R. Grinnell
E. Moore

Date: February 27, 2013
Time: 5:00 pm
Duration: 110 minutes

(Print) Surname: Solutions

(Print) Given Name(s): ______________________

Student Number: ______________________

Signature: ______________________

Circle the name of your Teaching Assistant and Tutorial Number:

Danny CAO 17 18	12 Peishan WANG
Fazle CHOWDHURY 21	6 Yimin WANG
Rui (Ray) GAO 11 19	14 23 Hoi Suen WONG
Angha GUPTA 15	7 Kevin YAN
Daniel MOGHBEL 4	20 Jianhao (Philip) YANG
Dongyuan (Sheldy) SHEN 10 16	2 3 Biyun (Ric) ZHANG
	25 Jingshun (Jason) ZHANG

Read these instructions:

1. This test has 11 pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and clearly indicate the location of your continuing work.
3. With the exception of the Multiple Choice questions, full points are awarded only for solutions that are correct, complete, and sufficiently display concepts and methods of MATA33.
4. You may use **one** standard calculator that does **not** perform any: graphing, matrix operations, numerical/symbolic differentiation or integration. Cell/smart phones, i-pods, other electronic devices, extra paper, notes and textbooks are forbidden at your workspace.
5. Cell/smart smart phones must be turned off and left at the front of the test room.
6. You are encouraged to write in pen or other ink, not pencil. Tests written in pencil will be denied any regrading privilege.

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Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6	7
F	A	E	B	E	C	D

Do not write anything in the boxes below.

Info.	Part A
2	21

Part B

1	2	3	4	5	6
16	15	14	14	11	7

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 3 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations and rough work.

1. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$ and $C = [C_{ij}] = 5A^TB^{-1}$ then $C_{11} - C_{22}$ is

   (A) -55 (B) 55 (C) 30 (D) -30 (E) 70 (F) 80

$$B^{-1} = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}$$

$$C = 5\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}$$

$$= 5\begin{bmatrix} 5 & -17 \\ 4 & -11 \end{bmatrix}$$

$$\boxed{\therefore\ C_{11} - C_{22} = 5(16) = 80}$$

2. If A is a 2×2 matrix such that $det(A) = 4$, then the value of $det\big((2A)^{-1}\big)$ is

   (A) 1/16 (B) 1/8 (C) 1/2 (D) 1 (E) none of (A) - (D)

$$det\left((2A)^{-1}\right) = \frac{1}{det(2A)} = \frac{1}{2^2\,det(A)} = \frac{1}{(4)(4)} = \boxed{\frac{1}{16}}$$

3. The maximum value of $Z = x - 3y$ subject to $1 \leq x \leq 5$ and $\frac{1}{2}x - \frac{1}{2} \leq y \leq -\frac{1}{2}x + \frac{9}{2}$ is

   (A) -1 (B) -2 (C) 2 (D) 13 (E) 1 (F) none of (A) - (E)

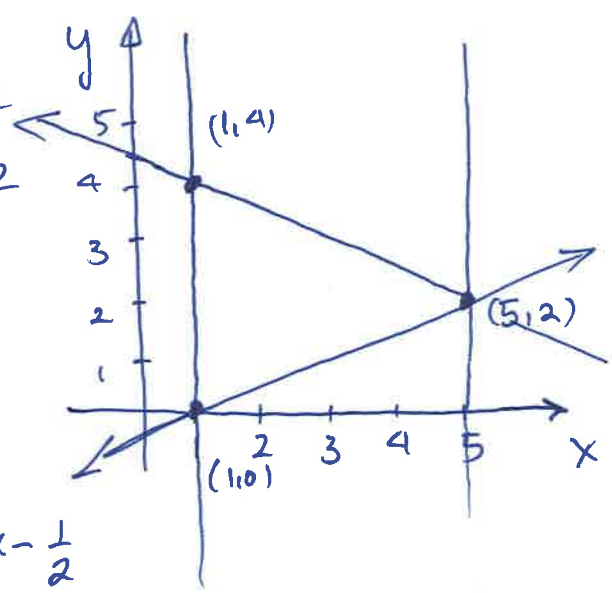
$Z(1,4) = -11$

$Z(5,2) = -1$

$\boxed{Z(1,0) = 1}$

$y = -\frac{1}{2}x + \frac{9}{2}$

$y(1) = 4$ $y(5) = 2$



$y = \frac{1}{2}x - \frac{1}{2}$

$y(1) = 0$ $y(5) = 2$

4. For what value(s) of the real constant c does the system of equations $\begin{aligned} x + y &= -0.5 \\ 4x + c^2 y &= c \end{aligned}$ have infinitely many solutions?

(A) 0 (B) -2 (C) 2 (D) (A) and (B) (E) (B) and (C) (F) no values of c

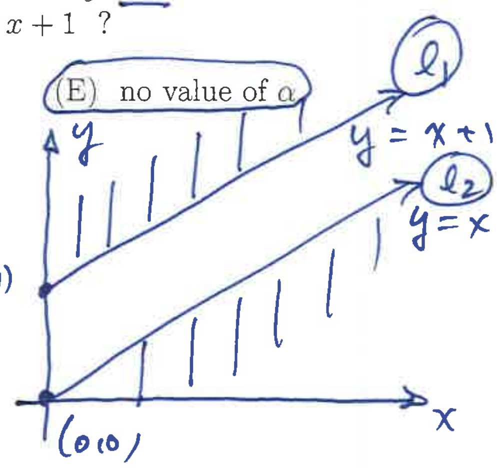
$$\left(\begin{array}{cc|c} 1 & 1 & -1/2 \\ 4 & c^2 & c \end{array}\right) \quad R_2 - 4R_1 \to R_2 \quad \left(\begin{array}{cc|c} 1 & 1 & -1/2 \\ 0 & c^2-4 & c+2 \end{array}\right)$$

∞ many solutions $\Longleftrightarrow$ $c^2 - 4 = 0$ and $c + 2 = 0$

$\boxed{\therefore\ c = -2}$

5. For what value of the real constant α does the function $Z = \alpha x - \alpha y$ not have a minimum value subject to the three constraints $x \geq 0,\ y \geq x,$ and $y \leq x + 1$?

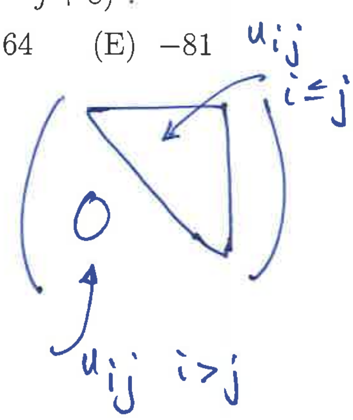
(A) 1 (B) -1 (C) 2 (D) a number not in (A) - (C) (E) no value of α



$$Z = \alpha(x - y) = \begin{cases} -\alpha & \text{on } l_1 \\ 0 & \text{on } l_2 \end{cases}$$

$\therefore$ (A) ✗ (B) ✗ (C) ✗ (D) ✗

6. Let $U = [u_{ij}]$ be the 8×8 upper triangular matrix such that $u_{ij} = -(i - j + 8)^2$. The largest element in U is (A) -4 (B) -1 (C) 0 (D) -64 (E) -81



$$u_{ij} = \begin{cases} 0 & \text{if } i > j \\ < 0 & \text{if } i \leq j \end{cases} \text{ iff } i \leq j,\ (i - j) \neq -8$$

7. Exactly how many of the following statements are always true?

✓ • A linear objective function defined on a non-empty, bounded, standard feasible region has a maximum value at some corner point.

✗ • If A and B are 2×2 matrices, then $A^2 - B^2 = (A - B)(A + B)$. Only if $AB = BA$

✗ • A system of two linear equations in three variables is consistent. $\begin{cases} x + y + z = 0 \\ x + y + z = 1 \end{cases}$

✗ • If P is a 2×2 matrix such that $P^2 = 0$, then $P = 0$. $P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

(Be sure you have printed the Multiple Choice answers in the boxes on page 2)

Part B - Full Solution Problem Solving

1. (a) Use the method of reduction to solve the linear system and display the reduced form of the augmented matrix. [8 points]

$$x_1 + 6x_2 + 0x_3 + 0x_4 + 4x_5 = -2$$
$$2x_1 + 12x_2 + x_3 + 0x_4 + 11x_5 = -3$$
$$-6x_1 - 36x_2 - x_3 + x_4 - 22x_5 = 13$$

$$\left(\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 2 & 12 & 1 & 0 & 11 & -3 \\ -6 & -36 & -1 & 1 & -22 & 13 \end{array}\right)$$

$$\begin{array}{r} R_2 - 2R_1 \to \\ R_3 + 6R_1 \to \end{array} \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1 & 1 & 2 & 1 \end{array}\right)$$

$$R_3 + R_2 \to \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right)$$

Reduced

Solution:

$x_1 = -2 - 6r - 4s$
$x_2 = r$
$x_3 = 1 - 3s$
$x_4 = 2 - 5s$
$x_5 = s$ $\quad r, s \in \mathbb{R}$ (parameters)

(b) Let a, b, and c be real constants and let ♡ represent the linear system [8 points]

$$x + 2y = a$$
$$-x - y = b$$
$$3x + 5y = c$$

Assume ♡ has a unique solution. Find the reduced form of the augmented matrix for ♡ and use this to express c in terms of a and b. Find the unique solution.

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ -1 & -1 & b \\ 3 & 5 & c \end{array}\right) \begin{array}{c} R_2 + R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array} \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b+a \\ 0 & -1 & c-3a \end{array}\right)$$

$$R_2 + R_3 \to \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & a+b \\ 0 & 0 & c-2a+b \end{array}\right) (R_1 - 2R_2) \to R_1 \left(\begin{array}{cc|c} 1 & 0 & -a-2b \\ 0 & 1 & a+b \\ 0 & 0 & c-2a+b \end{array}\right)$$

∵ ♡ is assumed to have a unique solution, we must have that $c - 2a + b = 0$, so

$$\boxed{c = 2a - b, \quad x = -a - 2b, \quad y = a + b}$$

2. Optimize $\boxed{Z = x+4y}$ subject to the constraints: $x+2y \geq 5$, $x \leq 3y$, $4x+5y \leq 34$, $x \geq 1$, and $y \leq 6$: Your solution should clearly show the feasible region, labeled corner points, the optimal value of Z, and all point(s) where the optimum occurs. [1 points]

L_1 is $x+2y=5$ Intercepts $(5,0)$, $(0, \frac{5}{2})$

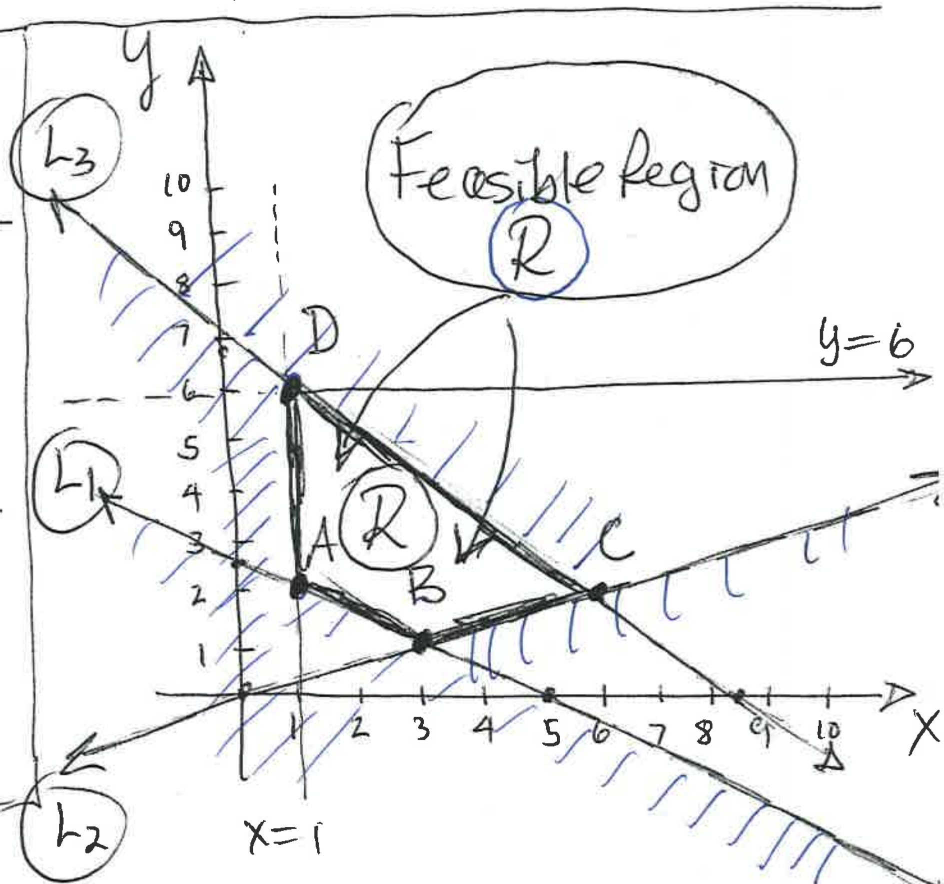
L_2 is $x+3y$ Intercept $(0,0)$, Extra pt. $(3,1)$

L_3 is $4x+5y=34$ Intercepts $(\frac{17}{2}, 0)$, $(0, \frac{34}{5})$

L_1 crosses $x=1$ line @ $(1,2)$

L_2 crosses $x=1$ line @ $(1, \frac{1}{3})$ & crosses L_1 @ $(3,1)$

L_3 crosses $x=1$ line @ $(1,6)$ and crosses L_2 @ $(6,2)$



Corner points for the feasible region R:

$A=(1,2)$ $B=(3,1)$ $C=(6,2)$ $D=(1,6)$

R is $\neq \phi$ bounded, and standard.

FTLP $\Rightarrow$ Z is minimized at a corner point.

$Z(A)=9$ $Z(B)=7$ $Z(C)=14$ $Z(D)=25$

MAX VALUE $= 25$ @ $D=(1,6)$

MIN VALUE $= 7$ @ $B=(3,1)$

FTLP = Fundamental Theorem of Linear Programming

REMARKS ① 2 points for citing $\neq \phi$, bded & corner pt via FTLP | ② 3 points for all features on diagram

3. (a) Find the inverse of the matrix $A = \begin{bmatrix} 0 & -7 & -2 \\ 0 & 5 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ [8 points]

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & -7 & -2 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & -7 & -2 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{matrix} -1R_1 \to R_1 \\ 1/5\, R_2 \to R_2 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/5 & 0 \\ 0 & -7 & -2 & 1 & 0 & 0 \end{array}\right]$$

$$R_3 + 7R_2 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & -2 & 1 & 7/5 & 0 \end{array}\right]$$

$$-\tfrac{1}{2} R_3 \to R_3 \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/5 & 0 \\ -1/2 & -7/10 & 0 \end{bmatrix}}_{A^{-1}}$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/5 & 0 \\ -1/2 & -7/10 & 0 \end{bmatrix}}$$

(b) Find all real x for which the matrix $\underbrace{\begin{bmatrix} -x & 0 & 0 \\ 0 & -6-x & -2 \\ 0 & 5 & 1-x \end{bmatrix}}_{\text{Call this } A(x)}$ is invertible. [6 points]

$A(x)$ is invertible $\iff \det(A(x)) \neq 0$

$$\det(A(x)) = (-x) \det \begin{bmatrix} -6-x & -2 \\ 5 & 1-x \end{bmatrix}$$

$$= (-x)\left[(-6-x)(1-x) + 10\right]$$

$$= (-x)\left[x^2 + 5x + 4\right] = (-x)(x+4)(x+1)$$

$$\boxed{\therefore A(x) \text{ is invertible} \iff x \neq -4, -1, 0}$$

4. (a) For each real number t, let $E(t)$ represent the system of two linear equations in the variables x and y:

$$tx + y = 4$$
$$2x + ty = 3 + tx$$

Find all values of t for which $E(t)$ has a unique solution and then use Cramer's rule to find the unique solution. [7 points]

Re-write $E(t)$: $tx + y = 4$

$(2-t)x + ty = 3$

Let $M(t) = \begin{pmatrix} t & 1 \\ 2-t & t \end{pmatrix}$

$E(t)$ has a unique solution $\iff \det(M(t)) \neq 0$

$\det(M(t)) = t^2 + t - 2$

$= (t+2)(t-1)$

By Cramer's rule:

$\boxed{x} = \dfrac{\det\begin{pmatrix} 4 & 1 \\ 3 & t \end{pmatrix}}{\det(M(t))} = \boxed{\dfrac{4t-3}{t^2+t-2}}$

$\boxed{y} = \dfrac{\det\begin{pmatrix} t & 4 \\ 2-t & 3 \end{pmatrix}}{\det(M(t))} = \boxed{\dfrac{7t-8}{t^2+t-2}}$

x, y defined only when

$\therefore E(t)$ has a unique solution $\iff t \neq -2, 1$

(b) Assume $P^{-1}AP = D$ where $P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, and $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Find A^k where k is an arbitrary positive integer. [7 points]

$$A = PDP^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$\therefore A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})(PDP^{-1})$$

$$= PD^kP^{-1}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2^k & -2^k \\ -2 & 3 \end{pmatrix}$$

$$\boxed{A^k = \begin{pmatrix} 3(2^k) - 2 & -3(2^k) + 3 \\ 2^{k+1} - 2 & -2^{k+1} + 3 \end{pmatrix}}$$

5. A small mining company operates two mines for the purposes of extracting gold and silver. The Saddle Mine costs $100,000 per day to operate, and it yields 5 kg of gold and 30 kg of silver each day. The Horseshoe Mine costs $150,000 per day to operate, and it yields 7.5 kg of gold and 10 kg of silver each day. Company management has set a target of at least 65 kg of gold and 180 kg of silver. How many days should each mine be operated so that the target can be met at a minimum cost? What is the minimum cost? Full points are awarded for a complete solution including a feasible region and appropriate details. [11 points]

Notes: ANSWER: (4,6), (7,4), (10,2) + (13,0)

(i) For your final answer only, "days" are to be interpreted as non-negative integer quantities.

(ii) You may assume there is a minimum cost, so you need not justify this.

Let x = # of days the Saddle Mine (SM) operates

y = # " " " Horseshoe " (HM) "

$x, y \in \mathbb{R}$, $x, y \geq 0$

Assume all money is in units of $1,000

Cost function $C = 100x + 150y$

"Gold constraint": $5x + 7.5y \geq 65$ ← ①

"Silver " ": $30x + 10y \geq 180$ ← ②

①: $2x + 3y \geq 26$ $(13,0), (0, 20/3)$

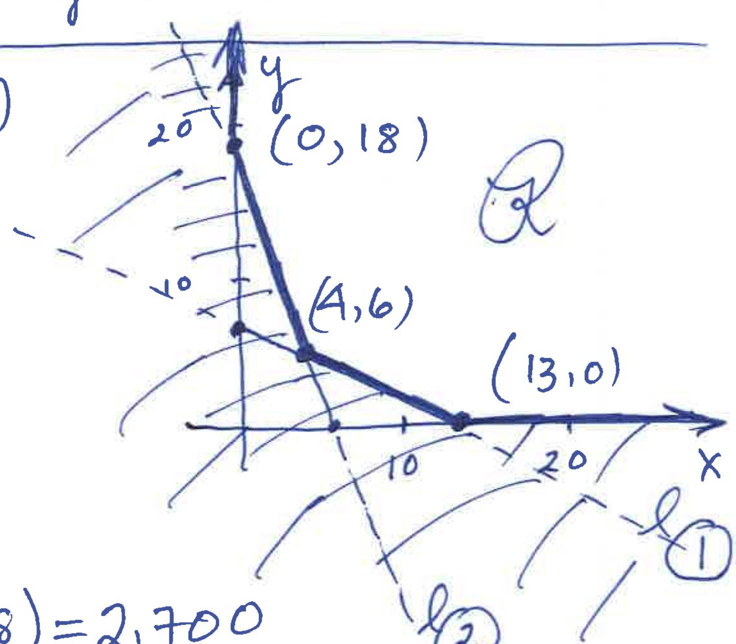
②: $3x + y \geq 18$ $(6,0), (0,18)$

Intersection: $y = 18 - 3x$

$2x + 3(18-3x) = 26$

$7x = 28 \to x = 4$

$y = 6$



R is the unbounded feasible region with corner points: $(0,18), (4,6), + (13,0)$

By (ii) above, MIN occurs @ a cornerpt!

$C(0,18) = 2,700$

$C(4,6) = 1,300$

$C(13,0) = 1,300$

By LP theory + (i), we look @ non-neg integer pts on $2x + 3y = 26$

They are:

x	4	7	10	13
y	6	4	2	0

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix, $n \geq 2$. Assume there are n non-zero real numbers $c_1,\ c_2,\ c_3, \ldots,\ c_n$ such that for each $j = 1,\ 2,\ 3,\ \ldots, n$ [7 points]

$$c_1 a_{1j} + c_2 a_{2j} + c_3 a_{3j} + \ldots + c_n a_{nj} = 0 \qquad ☺$$

Prove that A is not invertible.

Solution #1

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & & \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix}$$

Let $C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ We see from ☺ that $A^T C = 0$

so C is a non-trivial solution to the square homogeneous system $A^T X = 0$.

$\therefore$ by MATA33 theory, A^T is not invertible.

It follows that A is also not invertible. ∎

Solution #2 We use properties of determinant and ERO's to show that $\det(A) = 0$. This will imply A is not invertible.

$$\det(A) = \frac{1}{c_1 \cdot c_2 \cdot c_3 \cdots c_n} \det \begin{pmatrix} c_1 a_{11} & c_1 a_{12} & c_1 a_{13} & \cdots & c_1 a_{1n} \\ c_2 a_{21} & c_2 a_{22} & c_2 a_{23} & \cdots & c_2 a_{2n} \\ c_3 a_{31} & c_3 a_{32} & c_3 a_{33} & \cdots & c_3 a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \underbrace{c_n a_{n1}}_{j=1} & \underbrace{c_n a_{n2}}_{j=2} & \underbrace{c_n a_{n3}}_{j=3} & \cdots & \underbrace{c_n a_{nn}}_{j=n} \end{pmatrix}$$

$$= \frac{1}{c_1 \cdot c_2 \cdot c_3 \cdots c_n} \det \begin{pmatrix} \text{same } n-1 \text{ rows as above} \\ \underbrace{0 \quad 0 \quad 0 \quad \cdots \quad 0}_{\text{Used ☺ so that } R_n \text{ is created by } (R_n + R_1 + R_2 + \cdots + R_{n-1}) \text{ "add down" \& use ☺}} \end{pmatrix}$$

$= 0$

$\therefore$ A is not invertible ∎

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Basic Statistics

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$N = 528$ = # of students who wrote the test.

$\bar{X}$ = average $\approx 64.8\%$ ⊛

% passing (i.e. $\geq 50\%$) $\approx 86\%$ (Very good)

% $\geq 60\% \approx 65\%$ $\geq 80\% \approx 15\%$

% $\geq 70\% \approx 39.2\%$

	# of students	≈ % of students
90's	9	1.7
80's	71	13.4
70's	127	24.1
60's	137	26.0
50's	110	21.0
40's	54	10.0
30's	15	2.8
20's	5	1.0
10's	0	0
1's	0	0

⊛ All statistics are compiled before regrading.