*** SOLUTIONS ***

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiners: R. Grinnell
E. Moore

Date: February 18, 2012
Time: 9:00 am
Duration: 120 minutes

Provide the following information:

(Print) Surname: SOLUTIONS and STATISTICS

(Print) Given Name(s): ______________________

Student Number: ______________________

Signature: ______________________

Tutorial Number (e.g. TUT0033): ______________________

Carefully circle the name of your Teaching Assistant:

Rafayel BHUIYAN	Yinzheng (Jerry) GU	Allan MENEZES
Shibing CHEN	Anthony LEUNG	Daniel MOGHBEL
Christopher CHOW	Pourya MEMARPANAHI	Nancy TEMRAZ

Read these instructions:

1. This test has 11 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and indicate clearly the location of your continuing work.
3. With the exception of the Multiple Choice questions, full points are awarded only for solutions that are correct, complete, and sufficiently display concepts and methods of MATA33.
4. You may use **one** standard hand-held calculator (graphing facility is permitted). The following are forbidden: laptop computers, Blackberrys, cell-phones, I-Pods, MP-3 players, extra paper, textbooks, or notes.
5. You are encouraged to write in pen or other ink, not pencil. Tests written in pencil will be denied any regrading privilege.

*** SOLUTIONS ***

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6
A	D	E	D	C	E

Do not write anything in the boxes below.

Info.	**Part A**
2	24

Part B

1	2	3	4	5
14	18	16	14	12

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 4 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations and rough work.

1. If $A = \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ and $C = [C_{ij}] = BA - A^T$ then $6C_{11} - 4C_{22} + 2C_{12}$ is

(A) 6 (B) 4 (C) -6 (D) -8 (E) 12 (F) none of (A) - (E)

$BA - A^T = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 7 & -7 \\ 10 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 6 & -11 \\ 12 & 2 \end{bmatrix}$

$6(6) - 4(2) + 2(-11)$
$= 36 - 8 - 22$
$= 6$

2. For what real value(s) of x is the matrix $\begin{bmatrix} x-1 & 3 \\ 2 & x \end{bmatrix}$ not invertible ?

(A) all real x (B) 3 and 2 (C) -3 and 2 (D) 3 and -2 (E) none of (A) - (D)

$\det \begin{bmatrix} x-1 & 3 \\ 2 & x \end{bmatrix}$

Not invertible $\Leftrightarrow$ $\det = 0$

$\Leftrightarrow$ $x = 3$, $x = -2$

$= (x-1)x - 6$

$= x^2 - x - 6$

$= (x-3)(x+2)$

3. The maximum value of $Z = x - 3y$ subject to $-1 \leq y \leq 2$, $x \leq 2$, and $x - y \geq -2$ is

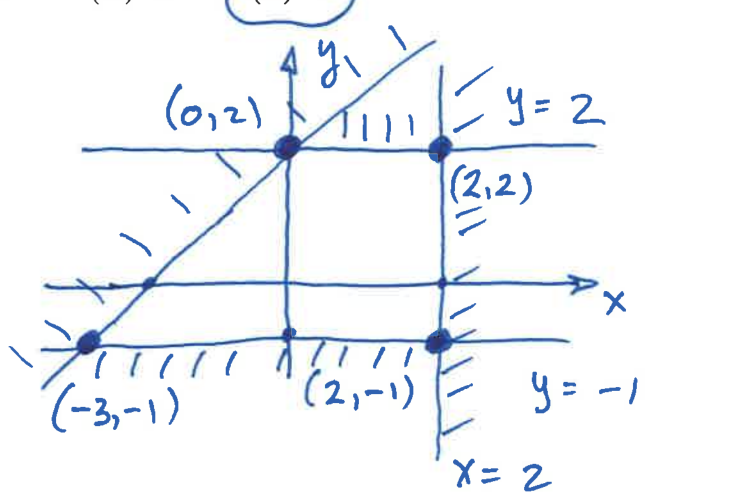
(A) none of (B) - (E) (B) 8 (C) 7 (D) 6 (E) 5

$Z(-3,-1) = -3 + 3 = 0$

$Z(0,2) = 0 - 6 = -6$

$Z(2,2) = 2 - 6 = -4$

$Z(2,-1) = 2 + 3 = 5$



4. Let r be a real constant. The system of equations $\begin{aligned} x-2y&=3 \\ 3x+y&=3 \\ x-9y&=r^2 \end{aligned}$ has a unique solution

(A) for all values of r (B) only when $r=0$ (C) only when $r=\pm 1$

(D) only when $r=\pm 3$ (E) for no values of r

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & 1 & 3 \\ 1 & -9 & r^2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 7 & -6 \\ 0 & -7 & r^2-3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 7 & -6 \\ 0 & 0 & a \end{array}\right)$$

$a = r^2 - 9$ must $= 0$ $\therefore$ $r = \pm 3$

5. Suppose M and G are 2×2 matrices such that $G^2 = M$. Consider the following mathematical statements:

✗ (i) If $M=0$ then $G=0$ ✓ (ii) $MG = GM$

✗ (iii) All entries in M are ≥ 0 ✓ (iv) If G is invertible, then M is also invertible.

Exactly how many of these statements are always true?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

(i) $\underset{G}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{G}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \underset{M}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$ (iii) $\underset{G}{\begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}} \underset{G}{\begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}} = \underset{M}{\begin{pmatrix} 0 & -3 \\ 0 & 9 \end{pmatrix}}$

(ii) $MG = G^2 G = G G^2 = GM$ (iv) $\det(G) \neq 0 \Rightarrow \det(M) \neq 0$

6. Let $Z = ax+by$ where a and b are non-zero constants. Let $\mathcal{R}$ be the feasible region consisting of all points in and on the triangle with corner points $(0,0)$, $(1,3)$, and $(3,3)$. Which one of the following statements gives the greatest amount of correct information about Z and $\mathcal{R}$?

✓ (A) Z has a maximum value at some corner point of $\mathcal{R}$

✗ (B) The minimum value of Z must occur at $(0,0)$

✓ (C) There exists a and b such that Z is maximized at every point on some edge of $\mathcal{R}$

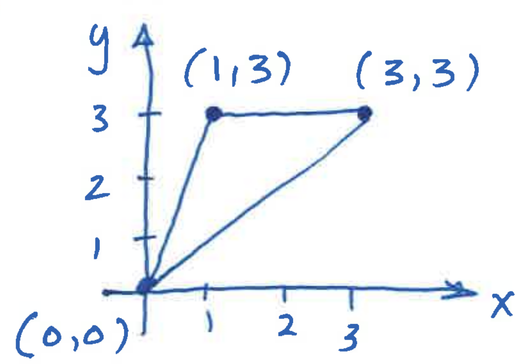
✗ (D) (A) and (B) are true, but (C) is false (E) (A) and (C) are true, but (B) is false

✗ (F) (A), (B), and (C) are true

(A) ✓ by FTLP

(B) ✗ consider $Z = x - y$

(C) ✓ consider $Z = x - y$



(Be sure you have printed the Multiple Choice answers in the boxes on page 2)

Part B - Full Solution Problem Solving

1. For each system of equations below, use the method of reduction to find the solution or determine that it is inconsistent. If a system is consistent, show the reduced form of its augmented matrix.

(a) [6 points]

$$\begin{aligned} x + 3y + z &= 8 \\ -3x + 3y - 3z &= -6 \\ -x + 9y - z &= 12 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ -3 & 3 & -3 & -6 \\ -1 & 9 & -1 & 12 \end{array}\right)$$

$R_2 + 3R_1 \to R_2$, $R_3 + R_1 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 0 & 12 & 0 & 18 \\ 0 & 12 & 0 & 20 \end{array}\right)$$

$R_3 - R_2 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 0 & 12 & 0 & 18 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

The 3rd row corresponds to the equation $0x + 0y + 0z = 2$ which has no solution.

Thus the system has no solution — it is inconsistent.

(b) [8 points]

$$\begin{aligned} x_1 - x_2 + 0x_3 + 5x_4 &= 0 \\ 2x_1 - 2x_2 + 3x_3 + 7x_4 &= 0 \\ 3x_1 - 3x_2 + 6x_3 + 9x_4 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & -1 & 0 & 5 \\ 2 & -2 & 3 & 7 \\ 3 & -3 & 6 & 9 \end{pmatrix}$$

$R_2 - 2R_1 \to R_2$, $R_3 - 3R_1 \to R_3$

$$\begin{pmatrix} 1 & -1 & 0 & 5 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 6 & -6 \end{pmatrix}$$

$\frac{1}{3}R_2 \to R_2$

$$\begin{pmatrix} 1 & -1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 6 & -6 \end{pmatrix}$$

$R_3 - 6R_2 \to R_3$

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{Reduced!}}$$

x_1, x_3 are basic variables

$x_2 = r$, $x_4 = s$ are free

$x_1 = x_2 - 5x_4$

$x_3 = x_4$

System is consistent.

It has infinitely many solutions.

The solutions are of the form:

$x_1 = r - 5s$

$x_2 = r$

$x_3 = s$

$x_4 = s$

$r, s \in \mathbb{R}$

2. Find the maximum and minimum values of the function $Z = -5x + 10y$ subject to the constraints: $-x + y \leq 3$, $-x + 6y \leq 13$, $5x - y \leq 22$, $x \geq -3$, and $y \geq -2$.

Your solution should clearly show the feasible region, labeled corner points, the optimum values of Z, and all point(s) where the optimum values occur. [18 points]

Lines $\ell_1 : -x + y = 3$ $\quad$ $\ell_4 : x = -3$

$\ell_2 : -x + 6y = 13$ $\quad$ $\ell_5 : y = -2$

$\ell_3 : 5x - y = 22$

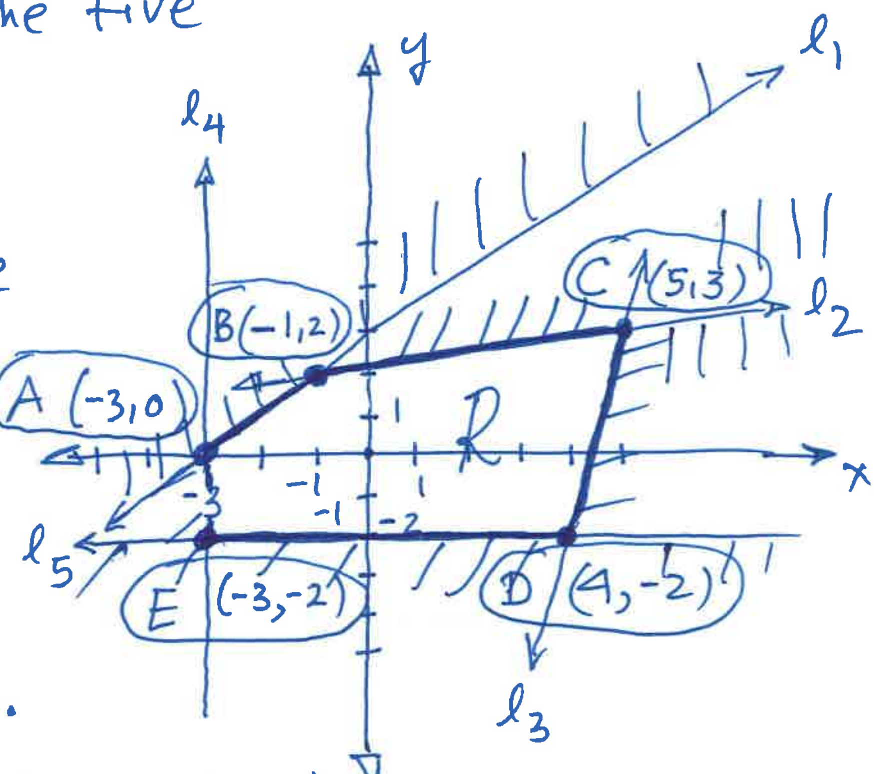
$\ell_1 \cap \ell_2$: $5y = 10$ $\therefore y = 2$. In ℓ_1: $-x + 2 = 3$ $\therefore x = -1$. Point $(-1, 2)$

$\ell_2 \cap \ell_3$: $29y = 87$ $\therefore y = 3$. In ℓ_2: $-x + 18 = 13$, $x = 5$. Point $(5, 3)$

$\ell_3 \cap \ell_5$: $5x + 2 = 22$ $\therefore x = 4$. Point $(4, -2)$

$(0,0)$ satisfies each of the five constraints given.

$\therefore$ the feasible region R is non-empty and is the unshaded polygon with corner points A, B, C, D, E at the right.



R is bounded, as the diagram clearly shows.

By FTLP (Fund. Thm of Linear Prog.), Z is optimized on R and we need only look @ the corner points of R:

$Z(A) = 15$ $\quad$ $Z(D) = -40$

$Z(B) = 25$ $\quad$ $Z(E) = -5$

$Z(C) = 5$

$\boxed{\begin{array}{l} \text{Max } Z = 25 \\ @ \ B(-1, 2) \\ \text{Min } Z = -40 \\ @ \ D(4, -2) \end{array}}$

3. In all of this question let $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ $\quad I - A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C$

(a) Find $(I-A)^{-1}$ where I is the 3×3 identity matrix. Show all work and correct notation for your row operations. [10 points]

$[C \mid I]$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$R_2 + R_1 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$R_2 \leftrightarrow R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \end{array}\right]$

$R_3 - R_2 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right]$

$\frac{1}{2} R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$R_1 - R_3 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$

$\qquad\qquad\qquad\quad I \qquad\qquad\qquad C^{-1}$

$\therefore (I-A)^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$

(b) Solve the matrix equation $X = AX + B$ where $B = \begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix}$ [6 points]

$IX = AX + B$

$\to IX - AX = B$

$(I-A)X = B$

$\therefore X = (I-A)^{-1}B = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 7 \end{bmatrix}$

$\therefore$ The solution to the matrix equation is

$X = \begin{bmatrix} -3 \\ -8 \\ 7 \end{bmatrix}$

4. A contractor builds and sells three types of houses represented symbolically as:
h_1 = "bungalow", h_2 = "sidesplit", h_3 = "ranch". The contractor builds and sells these house types in five cities: $c_1, \ldots, c_5$.

Consider the 5×3 "revenue matrix" $R = [R_{i,j}]$ where
$R_{i,j}$ = the contractor's revenue in \$100 thousands from selling house type h_j in city c_i.

For example, if the revenue from selling a "ranch" in city c_4 is \$759,000 then $R_{4,3} = 7.59$

(a) State the matrices C and Q such that: [5 points]
(i) the entries in the product RQ^T are, for each city, the average revenue over the three house types and
(ii) the entries in CR are, for each house type, the average revenue over the five cities.

For (i) $\underset{5\times 3}{R}\ \underset{3\times 1}{Q^T} \Rightarrow Q$ has size 1×3

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

(Other acceptable answers: $Q = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
or $Q = \frac{100{,}000}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ This gives the units in RQ^T in \$100,000)

For (ii) $\underset{1\times 5}{C}\ \underset{5\times 3}{R} \Rightarrow C = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$
(or $C = 20{,}000\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$)

(b) State the matrix B such that the entries in the product RB reflect all of the following:
(i) h_1 prices increase by 4% in each of the five cities.
(ii) h_2 prices remain unchanged.
(iii) h_3 prices decrease by 3% in each of the five cities. [3 points]

$\underset{5\times 3}{R}\ \underset{3\times 3}{B} \Rightarrow$ Product has size 5×3, as required

$$B = \begin{bmatrix} 1.04 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & .97 \end{bmatrix}$$

Question 4 continued.

(c) In city c_1 the contractor hopes to build and sell x sidesplits (house type h_2) and y ranches (house type h_3). The actual construction (i.e building) cost for each sidesplit is 33% of its revenue in city c_1 and for each ranch is 35% of its revenue in city c_1. The builder can spend at most \$10 million to construct all of these two house types in city c_1 and desires a profit of at least \$12 million for selling these two house types in city c_1. Give the system of four linear inequalities that describe this situation.

Recall that Profit = Revenue - Cost. [6 points]

Revenue in c_1 of h_2 is $R_{1,2}$

" " c_1 of h_3 is $R_{1,3}$

In units of \$100,000, \$10 million → 100

\$12 " → 120

System of inequalities is :

(cost) $.33 R_{1,2}\, x + .35 R_{1,3}\, y \leq 100$

(profit) $.67 R_{1,2}\, X + .65 R_{1,3}\, y \geq 120$

(non-neg) $x \geq 0 \quad y \geq 0$

5. (a) How many matrices of the form $P = \begin{bmatrix} a & 1 \\ 1 & -c \end{bmatrix}$, where a and c are $\overbrace{\text{natural numbers}}^{>0}$, have the commuting property $PQ = QP$ where $Q = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$? Sufficiently justify your solution. [7 points]

$$PQ = \begin{bmatrix} a & 1 \\ 1 & -c \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 2a+1 & a-5 \\ 2-c & 1+5c \end{bmatrix}$$

$$QP = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} a & 1 \\ 1 & -c \end{bmatrix} = \begin{bmatrix} 2a+1 & 2-c \\ a-5 & 1+5c \end{bmatrix}$$

To have $PQ = QP$, we need $a-5 = 2-c$

$\therefore\ a = 7-c$ where $a, c \in \{1, 2, 3, \ldots\}$

Look at the possibilities :

c	1	2	3	4	5	6
a	6	5	4	3	2	1

$\boxed{\text{Answer : There are 6 matrices}}$

Question 5 continued.

(b) Let $n \geq 3$ and let A be a $2n \times 2n$ matrix such that for each row:

$$\left.\begin{array}{l}\text{(the sum of the entries in the odd numbered columns)} \\ = \text{(the sum of the entries in the even numbered columns).}\end{array}\right\} (*)$$

Show that A is not invertible. [5 points]

Consider the homogeneous system

$AX = O$ where

O is the $(2n) \times 1$ 0-matrix and

X is the $(2n) \times 1$ variable matrix.

Let $B = [b_{i1}]$ where

$$b_{i1} = \begin{cases} 1 & \text{if } i \text{ is even} \\ -1 & \text{if } i \text{ is odd.} \end{cases}$$

B is a $(2n) \times 1$ matrix and looks like

$$B = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ \vdots \\ -1 \\ 1 \end{bmatrix}$$

We have that $AB = O$ because of property $(*)$

$\therefore$ the homogeneous system $AX = O$ has a non-trivial solution — namely B.

This implies A cannot be invertible.

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Some Midterm Test Statistics

Average $\approx 72.9\%$

Pass % $\approx \left(\dfrac{528-15}{528}\right) \times 100 \approx 97.2$

(i.e. about 97.2% of the 528 students who wrote the test actually passed)

Approximate percentages of students earning the following deciles:

100 $\longrightarrow$ 0%
90's $\longrightarrow$ 5.5%
80's $\longrightarrow$ 25.9%
70's $\longrightarrow$ 33.5%
60's $\longrightarrow$ 20.3%
50's $\longrightarrow$ 11.9%
40's $\longrightarrow$ 2.3%
30's $\longrightarrow$ 0.4%
20's $\longrightarrow$ 0.2%
10's $\longrightarrow$ 0%
1's $\longrightarrow$ 0%

This is very good ☺