* Solutions *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: June 22, 2016
Time: 5:00 pm
Duration: 110 minutes

Provide the following information

Last Name (PRINT BIG) * Solutions * __________________

Given Name(s) (PRINT BIG) __________________

Student Number __________________

Signature __________________

Circle the name of your Teaching Assistant and Tutorial Number

Hashiam Kadhim 3 Michael Moon 4

Jerry Shen 1 2 Ling-Sang Tse 5

Instructions

1. This test has 10 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and clearly indicate the location of your continuing work.
3. Make your answers correct and complete and show all of your work.
4. The following are forbidden at your workspace during any part of the test: calculators, smart phones, tablet devices, any kind of electronic transmission or receiving device, electronic dictionaries, extra paper, textbooks, notes, opaque (i.e. non-see through) pen/pencil cases, or food. You may have one drink, but it cannot be in a paper cup or box.
5. You are encouraged to write your test in pen or other ink, not pencil. If any portion of your test is written in pencil, your entire test will be denied any re-grading privilege.

* Solutions *

Do not write anything in the boxes below

Info	1	2	3	4	5	6	7	TOTAL
2	10	15	21	14	12	16	10	100

Questions begin here. Put your solutions and rough work in the answer spaces provided. Full-marks are awarded for answers that are correct, complete, and display a sufficient amount of relevant concepts from MATA33.

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1. Use the method of reduction to find the solution to the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 + 0x_4 - x_5 &= 2 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 4 \\ 0x_1 + 0x_2 + x_3 + x_4 + x_5 &= -1 \end{aligned}$$

[10 marks]

(Be sure to state the reduced form of the augmented matrix and show all ERO notation)

Augmented Matrix

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 2 \\ -1 & -1 & 2 & -3 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$R_2 + R_1 \to R_2 \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$-\tfrac{1}{3}R_2 \to R_2 \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$R_2 \leftrightarrow R_3 \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right)$$

$$R_1 + 2R_2 \to R_1 \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right)$$

$$R_1 - 2R_3 \to R_1 \underbrace{\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right)}_{\text{Reduced Form}}$$

- Leading 1's are circled
- Basic variables are x_1, x_3, x_4
- Free variables are x_2, x_5

$$\boxed{\begin{array}{l} \text{Final solution} \\ x_1 = 4 - r - s \\ x_2 = r \\ x_3 = 1 - s \\ x_4 = -2 \\ x_5 = s \\ r, s \in \mathbb{R} \text{ are parameters.} \end{array}}$$

2. In all of this question let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$.

(a) Find $det(A)$ by row or column expansion. [4 marks]

Expanding along row 3

$$det(A) = (1)(-1)^{1+3} \det\begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix} + (8)(-1)^{3+3} \det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$= 6 - 15 + 8(5-4)$$

$$= -9 + 8 = -1$$

(b) Find $det(A)$ using row operations to transform A to a triangular matrix. [3 marks]

$$det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= (1)(1)(-1) = -1$$

(c) Find A^{-1}. You need not show any ERO notation. [8 marks]

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Final answer is

$$\boxed{A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}}$$

3. In all of this question let $\mathcal{R}$ be the feasible region that is given by the five constraints:

$$x \geq -1, \qquad x - y \leq 6, \qquad -x + y \leq 3, \qquad 5x + 4y \leq 39, \qquad y \geq -1$$

(a) Find the maximum and minimum values of the linear objective function $Z = 5x + 2y$ where $(x, y) \in \mathcal{R}$.

(To earn full points, your solution must include a neat, labeled diagram of the feasible region $\mathcal{R}$, and all calculations/justifications. There is some extra answer space to this Part (a) at the top of the next page.)

[15 marks]

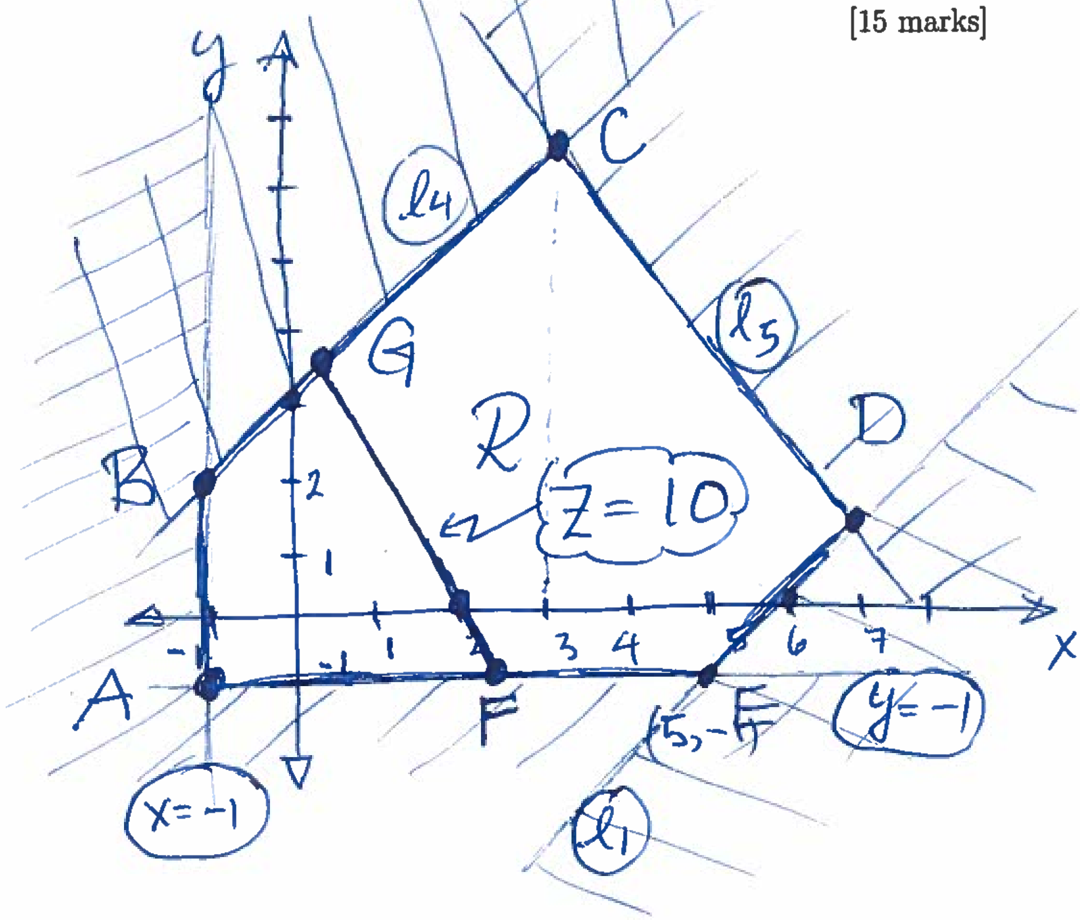
ℓ_1: $x - y = 6$

$(6,0)$, $(5,-1)$

ℓ_5: $5x + 4y = 39$

$$\left(\begin{array}{cc|c} 1 & -1 & 6 \\ 5 & 4 & 39 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 6 \\ 0 & 9 & 9 \end{array}\right)$$

$y = 1$, $x = 7$



ℓ_4: $-x + y = 3$

$(0,3)$, $(1,2)$, $(-1,2)$

$\ell_4 \cap \ell_5$:

$$\left(\begin{array}{cc|c} -1 & 1 & 3 \\ 5 & 4 & 39 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 3 \\ 0 & 9 & 54 \end{array}\right)$$

$y = 6$, $x = 3$

Corner points are:

$A = (-1,-1)$ $\quad D = (7,1)$

$B = (-1,2)$ $\quad E = (5,-1)$

$C = (3,6)$

$\mathcal{R}$ is the feasible region

It is $\neq \emptyset$, bounded, standard.

$\therefore$ optimize Z using FTLP

Answer continues

$Z = 5x + 2y$

Question 3 continues.

$Z(A) = -5 - 2 = -7$
$Z(B) = -5 + 4 = -1$
$Z(C) = 15 + 12 = 27$
$Z(D) = 35 + 2 = 37$
$Z(E) = 25 - 2 = 23$

Absolute Max is 37 @ $D = (7,1)$

Absolute Min is -7 @ $A = (-1,-1)$

(b) Let S be the set of points for which the objective function Z has value 10. Clearly indicate the set S in your diagram of $\mathcal{R}$ that you drew in Part (a). [2 marks]

See the level curve $Z = 10$ in **bold**
It has endpoints F + G

(c) Find the two linear objective functions Z_1 and Z_2 that both have an absolute maximum value of 33 at infinitely many points in $\mathcal{R}$. [4 marks]

Since there are two such objective functions, they must be maximized at all points on two parallel edges: ℓ_4 $-x + y = 3$
and ℓ_1 $x - y = 6$

Max along $\ell_4 = 33 \rightarrow -11x + 11y = 33$

Max along $\ell_1 = 33 \rightarrow \frac{11}{2}x - \frac{11}{2}y = 33$

$$\boxed{\begin{aligned} \therefore Z_1 &= -11x + 11y \\ Z_2 &= \frac{11}{2}x - \frac{11}{2}y \end{aligned}}$$

4. In all of this question, let $A = \begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 9 \\ 4 & 4 \end{pmatrix}$.

(a) Find the matrix D where $D = 3A(B^T)$. [3 marks]

$$D = 3\begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 9 & 4 \end{pmatrix} = 3\begin{pmatrix} 12 & 16 \\ 34 & 40 \end{pmatrix} = \begin{pmatrix} 36 & 48 \\ 102 & 120 \end{pmatrix}$$

(b) Express the columns of D as linear combinations of the columns of A. [4 marks]

$$\begin{pmatrix} 36 \\ 102 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 7 \end{pmatrix} + 27\begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 48 \\ 120 \end{pmatrix} = 12\begin{pmatrix} 3 \\ 7 \end{pmatrix} + 12\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(c) Find the matrix Q such that $AQ = B$. [4 marks]

$$Q = A^{-1}B = \frac{1}{\det(A)}\begin{pmatrix} 3 & -1 \\ -7 & 3 \end{pmatrix} B$$

$$= \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 1 & 9 \\ 4 & 4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 23 \\ 5 & -51 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{23}{2} \\ \frac{5}{2} & -\frac{51}{2} \end{pmatrix}$$

(d) Find a matrix C whose square is B, or show that there is no such matrix C. [3 marks]

Suppose C is a 2×2 matrix and $C^2 = B$

$\therefore \det(C^2) = \det(B) = -32$

But $\det(C^2) = (\det(C))^2$ so $(\det(C))^2 = -32$

which is impossible.

$\boxed{\therefore \text{there is no such matrix } C}$

5. A total of $\$1,000$ is invested amongst three separate funds: A, B, and C. The amount invested in fund B is 40% of that portion left after the amount of investment in A only has been made. The amount invested in fund A is the fraction $\left(\frac{5}{17}\right)^{ths}$ of that portion left after the amount of investment in B only has been made. (*)

(a) State a system of three linear equations in three variables with integer coefficients that represents the investment problem above. [5 marks]

x = amount invested in A
y = " " " B
z = " " " C
(all in \$)

For x & y, (*) says

$y = \frac{2}{5}(1000 - x)$

$x = \frac{5}{17}(1000 - y)$

System is

$$\boxed{\begin{aligned} x + y + z &= 1000 \\ 2x + 5y &= 2000 \\ 17x + 5y &= 5000 \end{aligned}}$$

(Coefficients are integers).

(b) Use the method of reduction to solve your system in Part (a) and state how much is invested in each of the three funds. (You need not show ERO notation and you need not produce the complete rref of the coefficient matrix in order to earn full marks.) [7 marks]

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1000 \\ 2 & 5 & 0 & 2000 \\ 17 & 5 & 0 & 5000 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1000 \\ 0 & 3 & -2 & 0 \\ 0 & -12 & -17 & -12000 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1000 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & -25 & -12000 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1000 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & 480 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 520 \\ 0 & 3 & 0 & 960 \\ 0 & 0 & 1 & 480 \end{array}\right)$$

$z = 480$, $y = \frac{960}{3} = 320$

1st row says

$x = 520 - 320 = 200$

Final answer

$$\boxed{\begin{aligned} x &= 200 \\ y &= 320 \\ z &= 480 \end{aligned}}$$

6. A factory makes two kinds of products, A and B. For a given work shift, each unit of A requires 4 hours of electronic work and 2 hours of assembly, and each unit of B requires 3 hours of electronic work and 1 hour of assembly. For each shift, there is a maximum of 240 hours of electronic work time available and 100 hours of assembly time available. Each unit of A earns a profit of \$7 and each unit of B earns a profit of \$5.

(a) Use methods of linear programming to find how many of A and B should be made per shift in order to maximize the total profit and satisfy all of the conditions above. What is the maximum profit? [12 marks]

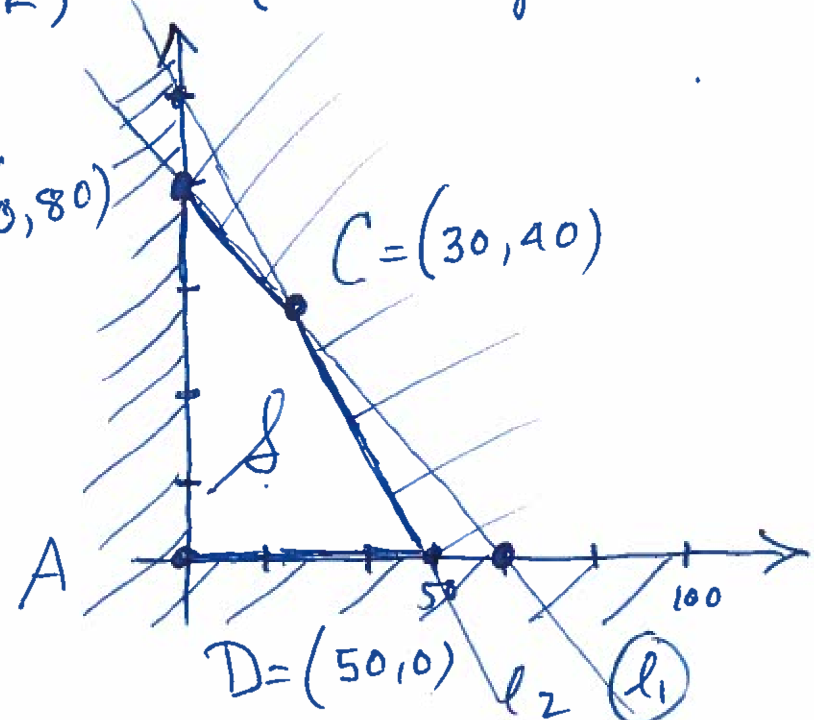
Let x = amount of A made per shift
y = " " B " " "

We maximize profit $P = 7x + 5y$ subject to
$x, y \geq 0$, $4x + 3y \leq 240$, (electric work) $2x + y \leq 100$ (assembly)

(l_1) $4x + 3y = 240$
$(60,0)$, $(0,80)$

(l_2) $2x + y = 100$
$(50,0)$, $(0,100)$

$l_1 \cap l_2$ $y = 40$
$x = 30$



(b) If the unit profit for A and B both increase by \$1 per unit, what (integer) numbers of A and B should be made per shift in order to maximize total profit? [4 marks]

Feasible set S is $\neq \phi$, bded, standard

By FTLP,
$P(A) = 0$
$P(B) = 400$
$P(C) = 210 + 200 = 410$
$P(D) = 350$

Max Profit is \$410 when $x = 30$, $y = 40$

For (b), $Q = 8x + 6y$
$Q(A) = 0$
$Q(B) = 480$
$Q(C) = 240 + 240 = 480$
$Q(D) = 400$

$\therefore$ Max Profit at all points (x,y) on segment BC where x is an integer.

7. In all of this question let A be an $n \times n$ matrix, $n \geq 2$, such that $A^{-1} = A$.

(a) An $n \times n$ matrix M is called an **idempotent** if and only if $M^2 = M$. Show that the matrix $\frac{1}{2}\left(A + I\right)$ is an idempotent. [5 marks]

Let $M = \frac{1}{2}(A+I)$. We show $M^2 \overset{?}{=} M$ under the assumption $A^{-1} = A$ (i.e. $A^2 = I$)

$M^2 = \frac{1}{2}(A+I)\frac{1}{2}(A+I)$.

$= \frac{1}{4}(A^2 + 2A + I)$.

$= \frac{1}{4}(I + 2A + I)$

$= \frac{1}{4}(2A + 2I) = \frac{1}{2}(A+I) = M$

$\therefore \frac{1}{2}(A+I)$ is an idempotent

(b) Consider the matrix equation $AX = B$ where X is an $n \times 1$ matrix of variables and B is an $n \times 1$ matrix of constants. If we assume all of the entries in A and B are integers only, show that the solution matrix X must have all of its entries as integers only.
(Hint: Cramer's rule) [5 marks]

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be the matrix of variables.

Assume $A^{-1} = A$ so $A^2 = I$ $\therefore (\det(A))^2 = \det(A^2) = 1$

$\therefore \det(A) \in \{-1, 1\}$

For each $j = 1, 2, \ldots, n$, Cramer's rule says

$x_j = \dfrac{\det(A_j(B))}{\det(A)} = \pm \det(A_j(B))$

All entries in $A_j(B)$ are integers and $\det(A_j(B))$ is computed with operations $+, -, \times$ only (no $\div$) $\therefore \det(A_j(B))$ is an integer, so x_j is also.