SOLUTIONS

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: June 22, 2012
Time: 7:00 pm
Duration: 120 minutes

Provide the following information:

(Print) Surname: SOLUTIONS

(Print) Given Name(s): (statistics on page 11)

Student Number: ____________________

Signature: ____________________

Tutorial Number (e.g. TUT0033): ____________________

Carefully circle the name of your Teaching Assistant:

Yuri CHER   Allan MENEZES   Vitaly KUZNETSOV

Carefully read these instructions:

1. This test has 11 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page, and indicate clearly the location of your continuing work.
3. Full points are awarded for Part B solutions only if they are correct, complete, and sufficiently display concepts and methods of MATA33.
4. You may use **one** standard hand-held calculator (graphing facility is permitted), but it cannot be able to perform any kind of matrix manipulations, differentiation, or integration. The following are forbidden: laptop computers, Blackberrys, cell-phones, I-Pods, MP-3 players, extra paper, textbooks, or notes.
5. You are encouraged to write in pen or other ink, not pencil. Tests written in pencil will be denied any regrading privilege.

Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6	7
B	C	D	C	C	A	B

(reasoning/calculations on p. 3&4)

Do not write anything in the boxes below.

Part A
21

Part B					
1	2	3	4	5	6
11	15	16	12	11	14

Total
100

Part A - Multiple Choice For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 3 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations and rough work.

1. If $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$ then $2AB - B^T$ equals

   (A) $\begin{bmatrix} 13 & -12 \\ 0 & 13 \end{bmatrix}$ (B) $\begin{bmatrix} 13 & -13 \\ 0 & 13 \end{bmatrix}$ (C) $\begin{bmatrix} 15 & -13 \\ 0 & 13 \end{bmatrix}$ (D) $\begin{bmatrix} 13 & 13 \\ 0 & -13 \end{bmatrix}$

   (E) none of (A) - (D)

$$2\begin{bmatrix}4&-3\\2&1\end{bmatrix}\begin{bmatrix}1&2\\-1&5\end{bmatrix}-\begin{bmatrix}1&-1\\2&5\end{bmatrix}$$

$$=2\begin{bmatrix}7&-7\\1&9\end{bmatrix}-\begin{bmatrix}1&-1\\2&5\end{bmatrix}=\begin{bmatrix}14&-14\\2&18\end{bmatrix}-\begin{bmatrix}1&-1\\2&5\end{bmatrix}$$

$$=\begin{bmatrix}13&-13\\0&13\end{bmatrix}$$

2. If A and B are as in Question 1 and $C = \begin{bmatrix} -1 & 8 \\ 1 & 4 \end{bmatrix}$ then the value of $det(AB + AC)$ is

   (A) 10 (B) −8 (C) 0 (D) 40 (E) none of (A) - (D)

$$\det(AB+AC)=\det(A(B+C))=\det(A)\det(B+C)$$

$$=(4+6)\det\begin{pmatrix}0&10\\0&9\end{pmatrix}=(10)(0)=0$$

3. The maximum value of $Z = 3x - y$ subject to the inequalities $0 \le y \le 2$, $x \le 2$, and $y \le x + 2$ is

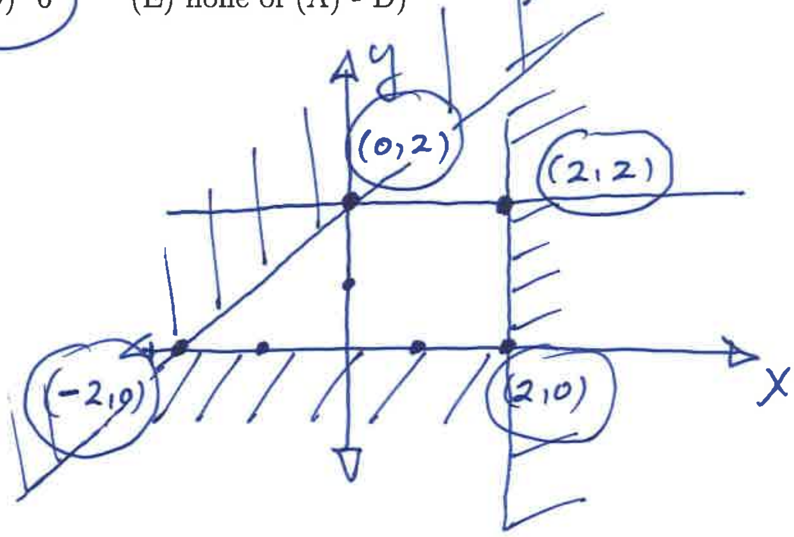
   (A) −2 (B) 4 (C) 5 (D) 6 (E) none of (A) - D)

$Z(-2,0) = -6$

$Z(0,2) = -2$

$Z(2,2) = 4$

$Z(2,0) = 6$ MAX



4. Exactly how many of the following five properties are mathematically equivalent to the statement: For a 3×3 matrix A, $det(A) \neq 0$?

X (*i*) The trivial solution is a solution to the matrix equation $AX = 0$.

✓ (*ii*) A has an inverse. X (*iii*) $CA = AC$ for some 3×3 matrix C.

✓ (*iv*) $det(A^2) > 0$. ✓ (*v*) The reduced form of A is the 3×3 identity matrix.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Properties (ii), (iv), and (v) are equivalent only.

5. Let $H = [h_{ij}]$ be the 8×8 lower triangular matrix where $h_{ij} = i^2 + 3j^2 - 5$ for all $i \geq j$. The smallest element in H is

(A) -3 (B) -2 (C) -1 (D) 0 (E) none of (A) - (D)

h_{ij} is smallest when i and j are smallest.

$\therefore h_{11} = 1 + 3 - 5 = -1$ = smallest entry in H.

6. If P and Q are 3×3 matrices such that $det(P) = -2$ and $det(Q) = 4$ then $det\left(\frac{det(Q)}{2} P\right)$ = D equals

(A) -16 (B) -64 (C) -4 (D) -1 (E) a number not in (A) - (D)

$D = det\left(\frac{4}{2} P\right) = det(2P) = 2^3 det(P) = 8(-2)$

$= -16$

7. Let $\mathcal{R}$ represent the region consisting of all points within and on the edges of the triangle whose vertices are $(0,0)$, $(0,1)$, and $(2,0)$. Let $Z = x + by$ where $b < 0$ is a real constant. Consider the following assertions:

X (i) Z is minimized at the point $(0,0)$ only.

X (ii) Z is minimized at every point on some edge of $\mathcal{R}$.

✓ (iii) Z is maximized at the point $(2,0)$ only.

✓ (iv) Z is minimized at the point $(0,1)$ only.

Which one of the following must be true ?

(A) (i) and (ii) (B) (iii) and (iv) (C) (ii) and (iii) (D) (iv)

(E) we cannot determine the truth of any of the statements above because the exact value of the constant b is not known.

(Be sure you have printed the Multiple Choice answers in the boxes on page 2)

Part B - Full Solution Problem Solving

1. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 4 & -2 \\ -4 & -6 & 1 \\ 3 & 5 & -1 \end{pmatrix}$

and check your answer by multiplication. [11 points]

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ -4 & -6 & 1 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ -4 & -6 & 1 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 2 & -3 & 2 & 1 & 0 \\ 0 & -1 & 2 & -3/2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & -2 & 3/2 & 0 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 3/2 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1/2 & 1 & 2 \\ 0 & 1 & 0 & -1/2 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -3 & -4 \\ 0 & 1 & 0 & -1/2 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

Check: $\begin{pmatrix} 2 & 4 & -2 \\ -4 & -6 & 1 \\ 3 & 5 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & -3 & -4 \\ -1/2 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

$$\therefore A^{-1} = \begin{pmatrix} 1/2 & -3 & -4 \\ -1/2 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}$$

2. For each of the following systems of linear equations, use the method of reduction to find the solution or determine that the system is inconsistent. If the system is consistent, be sure to display the reduced form of its augmented matrix.

(a)
$$\begin{aligned} 2x + y + 3z &= 0 \\ x - y + z &= 0 \\ -x - y + 4z &= 0 \end{aligned}$$
[7 points]

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\sim \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & -5/2 \end{pmatrix}} \sim \begin{pmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{reduced form}}$$

At this step we know the system is consistent

Solution is $x = y = z = 0$

(b)
$$\begin{aligned} x_1 + 2x_2 - 5x_3 + 6x_4 + 14x_5 &= 20 \\ 2x_1 + 4x_2 - 9x_3 + 12x_4 + 29x_5 &= 41 \\ -x_1 - 2x_2 + 5x_3 - 5x_4 - 12x_5 &= -20 \end{aligned}$$
[8 points]

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 6 & 14 & 20 \\ 2 & 4 & -9 & 12 & 29 & 41 \\ -1 & -2 & 5 & -5 & -12 & -20 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 6 & 14 & 20 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 6 & 14 & 20 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 6 & 19 & 25 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\sim \begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 7 & 25 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right) \\ \begin{array}{ccccc} x_1 & \underset{x_2}{r} & x_3 & x_4 & \underset{x_5}{s} \end{array} \end{array} = \text{reduced form}$$

x_1, x_3, x_4 are basic variables
$x_2 (=r), (x_5 = s)$ are free variables

Solution is
$$\begin{aligned} x_1 &= 25 - 2r - 7s \\ x_2 &= r \\ x_3 &= 1 - s \\ x_4 &= -2s \\ x_5 &= s \end{aligned}$$

r, s are parameters; $r, s \in \mathbb{R}$

3. Find the maximum and minimum values (and where they occur) for the objective function $Z = -2x + 8y$ subject to the five constraints:

$$x - 4y + 15 \geq 0, \qquad x \leq 5, \qquad x - 2y \leq 5, \qquad x + 3y \geq 0, \qquad y \leq 4x$$

(To earn full points, your solution must include a neat, labeled diagram of the feasible region, the location of all point(s) where Z is optimized, and all calculations/justifications)

[16 points]

ℓ_1 $\quad x - 4y + 15 = 0$

$(1,4)$ $\quad 1 - 16 + 15 = 0$ ✓

$(5,5)$ $\quad 5 - 20 + 15 = 0$ ✓

ℓ_2 $\quad x = 5$

$(5,5)$ ✓ $\quad (5,0)$ ✓

ℓ_3 $\quad x - 2y = 5$

$(5,0)$ $\quad 5 - 0 = 5$ ✓

$(3,-1)$ $\quad 3 + 2 = 5$ ✓

ℓ_4 $\quad x + 3y = 0$

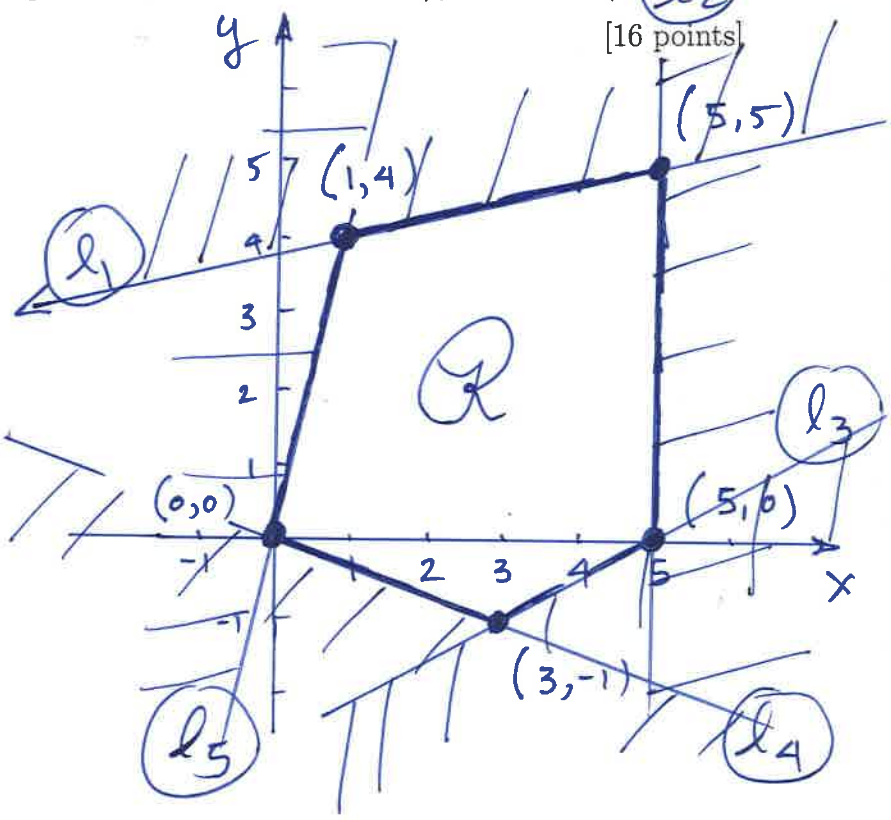
$(3,-1)$ $\quad 3 - 3 = 0$ ✓

$(0,0)$ $\quad 0 + 0 = 0$ ✓

ℓ_5 $\quad y = 4x$

$(0,0)$ $\quad 0 = 0$ ✓

$(1,4)$ $\quad 4 = 4$ ✓



R = feasible region
$= \{(x,y) \mid (x,y)$ satisfies all inequalities$\}$

R is clearly non-empty and bounded. By FTLP, Z is optimized via corner point evaluation

$Z(0,0) = 0$

$Z(1,4) = -2 + 32 = 30$

$Z(5,5) = -10 + 40 = 30$

$Z(5,0) = -10$

$Z(3,-1) = -6 - 8 = -14$

$\therefore$ MAX value of Z is 30 at every point on segment joining $(1,4)$ to $(5,5)$. MIN value is -14 at $(3,-1)$

4. An company manufactures four different models of desk lamps M_1, M_2, M_3, M_4 in three separate locations L_1, L_2, L_3. The manufacturing data is stored in the matrix

$$A = [a_{ij}] = \begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix}$$

where a_{ij} = the number of model M_j produced in location L_i in April 2012.

(a) State the matrices P, Q, and R such that [7 points]

(i) the entries in AP^T give the sum of the manufacturing outputs at each of the three locations in April 2012.

$\underset{3\times 4}{A}\ \underset{4\times 1}{P^T}$ ∴ P is 1×4 and "adds"

$P = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

(ii) the entries in QA give the average manufacturing output of each of the four models in April 2012.

$\underset{1\times 3}{Q}\ \underset{3\times 4}{A}$ ∴ Q is 1×3 and gives average

$Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

(iii) the entry in RAP^T is the total production of all models in all locations in April 2012.

$\underset{1\times 3}{R}\ \underset{3\times 4}{A}\ \underset{4\times 1}{P^T}$

$R = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

∴ R is 1×3 and "adds"

(b) The output in May 2012 compared to April 2012 saw a 5% increase in production of all models at location L_1 ; no change at location L_2 ; and a 10% decrease in all model production at location L_3. Find the matrix C such that CA shows all of the May 2012 production. Find the matrix D so that the matrix $A + D$ also shows all May 2012 production.

[5 points]

$$C = \begin{bmatrix} 1.05 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & .9 \end{bmatrix}$$

$$D = \begin{bmatrix} 16 & 14 & 23 & 14 \\ 0 & 0 & 0 & 0 \\ -54 & -42 & -20 & -88 \end{bmatrix}$$

5. Let $\mathcal{R}$ represent the feasible region defined by the three inequalities:

$$y \le x+2, \qquad x+y \ge 2, \qquad 2y \ge x-2.$$

Let $a > 0$ be a real constant and consider an objective function of the form $Z = ax + 2ay$. Show that Z does not have a maximum on $\mathcal{R}$ but does have a minimum on $\mathcal{R}$. Find where the minimum occurs and its value. Full points are awarded only if your solution shows a good sketch of $\mathcal{R}$ and sufficient justification and detail. [11 points]

ℓ_1 $\quad y = x+2$

$(0,2)$ $\quad 2 = 0+2$ ✓

$(1,3)$ $\quad 3 = 1+2$ ✓

ℓ_2 $\quad x+y=2$

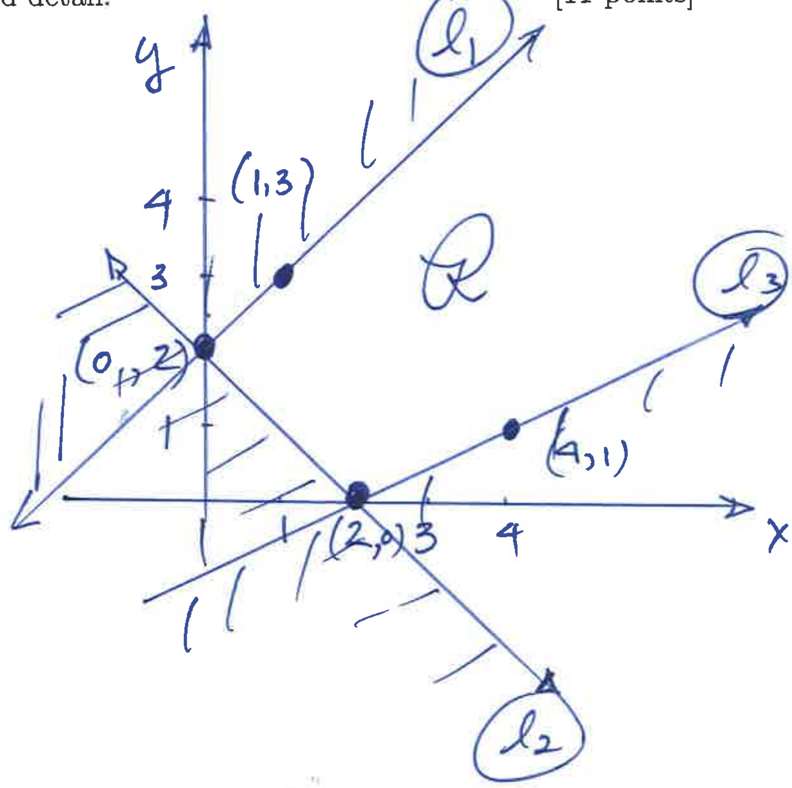
$(0,2)$ $\quad 0+2=2$ ✓

$(2,0)$ $\quad 2+0=2$ ✓

ℓ_3 $\quad 2y = x-2$

$(2,0)$ $\quad 0 = 2-2$ ✓

$(4,1)$ $\quad 4-2=2$ ✓



$\mathcal{R}$ = feasible region
$= \{(x,y) \mid (x,y)$ satisfies the 3 given inequalities$\}$

$\mathcal{R}$ is clearly non-empty and not bounded.

Z has no MAX on $\mathcal{R}$: The ray $y = x+2$, $x \ge 0$, is an edge of $\mathcal{R}$, so it is part of $\mathcal{R}$.
Restrict (x,y) to this ray to get $Z(x, x+2) = ax + 2a(x+2)$
$= 3ax + 4a \to \infty$ as $x \to \infty$

Z has a MIN on $\mathcal{R}$: Look @ level curve $Z = c$

$c = ax + 2ay \to y = -\frac{a}{2a}x + \frac{c}{2a} = -\frac{1}{2}x + \frac{c}{2a}$

This is parallel to the line $y = -\frac{1}{2}x + 1$ and touches $\mathcal{R}$ @ $(2,0)$ and at no point with smaller y-value.
$\therefore$ Z is MIN @ $(2,0)$ of value $Z(2,0) = 2a$.

6. (a) Suppose your annual income is A dollars. An amount of x dollars is paid in federal income tax and y dollars is paid in provincial income tax. The federal tax amount is 1/3 of the portion of your income remaining after the provincial tax has been paid. The provincial amount is 2/9 of the portion of your income remaining after the federal tax has been paid. Find the actual percentage of your income that is paid in federal tax and paid in provincial tax.

[8 points]

We have equations $x = \frac{1}{3}(A-y)$

$y = \frac{2}{9}(A-x)$

$\therefore \quad 3x + y = A \leftarrow ①$

$2x + 9y = 2A \leftarrow ②$

① gives $y = A - 3x$. Sub in ②

$2x + 9(A-3x) = 2A$

$2x + 9A - 27x = 2A$

$-25x = -7A$

$\therefore x = \frac{7}{25}A \; (= 28\%)$

$y = A - \frac{21}{25}A = \frac{4}{25}A \; (= 16\%)$

$\therefore$ percentage of income in federal tax is 28% and in provincial tax is 16%

(b) Let W be an arbitrary $n \times n$ matrix, $n \geq 2$, and let $U = WW^T$. If the diagonal entries in U are all 0, show that $W = 0$. [6 points]

$W = [w_{ij}]$ and $U = [u_{ij}]$

$u_{ij} = (i^{th}\text{ row of } W)(j^{th}\text{ column of } W^T)$

$= (i^{th}\text{ row of } W)(j^{th}\text{ row of } W)$

$\therefore u_{ii} = \sum_{k=1}^{n} w_{ik}^2 = w_{i1}^2 + w_{i2}^2 + \cdots + w_{in}^2$

$\because u_{ii} = 0 \Rightarrow w_{ij}^2 = 0$ for all $j = 1, \ldots, n$, hence $w_{ij} = 0$ for all $j = 1, \ldots, n$. But this is true for all $i = 1, \ldots, n$ too. $\therefore w_{ij} = 0$ all $i, j = 1, 2, \ldots, n$

$\therefore W = 0$.

~~(This page is intentionally left blank)~~

Test Statistics

96 students wrote the test.

~ 62.6% is the average.

~ 84.4% of students passed.

Approximate % of students in each decile

100	0
90's	3.1
80's	9.4
70's	16.7
60's	25.0
50's	30.2
40's	11.5
30's	4.2
20's	0
10's	0
1's	0