Solutions only ... not questions.

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33S - Calculus for Management II

Examiners: P. Glynn-Adey
R. Grinnell

Date: February 13, 2016
Time: 9:00 am
Duration: 2 hours

Provide the following information

Last Name (PRINT BIG) ______________________

Given Name(s) (PRINT BIG) ______________________

Student Number ______________________

Signature ______________________

Solutions only ... not questions.

Circle the name of your Teaching Assistant and Tutorial Number

Ayaan CHAUDHRY	26	Michael MOON	21 24
Ruixue DAI	6	Ke TONG	12
Taylor ESCH	13	Huiyi WANG	1
Rui GAO	5	Tianqi WANG	15
Terry (Yaodong) GAO	4	Dexter WU	16
Amir HEJAZI	25	Binya XU	23
Martin HO	7	Ruoqi YU	17
Pourya MEMARPANAHI	8	Elaine (Mengnan) ZHU	10

No Statistics!

Given:

(1) $B = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$ $C = \begin{pmatrix} -1 & 17 \\ -3 & 16 \end{pmatrix}$

(a) $3B + C^T = \begin{pmatrix} 3 & 12 \\ -3 & 9 \end{pmatrix} + \begin{pmatrix} -1 & -3 \\ 17 & 16 \end{pmatrix}$

$= \begin{pmatrix} 2 & 9 \\ 14 & 25 \end{pmatrix}$ [3 marks]

$\det(B) = 7$

$\det(\det(B)\, C) = \det(7C) = 7^2 \det(C)$ [4 marks]

$= (49)(35)$

$= 1{,}715$

(b) $BC = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 17 \\ -3 & 16 \end{pmatrix} = \begin{pmatrix} -13 & 81 \\ -8 & 31 \end{pmatrix}$

$\begin{pmatrix} -13 \\ -8 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 81 \\ 31 \end{pmatrix} = 17\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 16\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$= (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-3)\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ [7 marks]

(c) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Solve $AB = C$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 17 \\ -3 & 16 \end{pmatrix}$

$a - b = -1$ $4a + 3b = 17$

$c - d = -3$ $4c + 3d = 16$

We get $a = 2$ $b = 3$

$c = 1$ $d = 4$

Can also do:

$A = CB^{-1}$

$= \begin{pmatrix} -1 & 17 \\ -3 & 16 \end{pmatrix}\left(\frac{1}{7}\right)\begin{pmatrix} 3 & -4 \\ 1 & 1 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

[6 marks]

② (a) Work from augmented matrix & row reduce: [8 marks]

$$\left(\begin{array}{ccccc|c} -3 & -18 & -3 & 0 & -21 & 3 \\ 3 & 18 & 4 & 1 & 29 & 0 \\ 0 & 0 & 0 & -2 & -10 & -4 \end{array}\right) \begin{array}{l} R_2 + R_1 \to R_2 \\ -\frac{1}{2} R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} -3 & -18 & -3 & 0 & -21 & 3 \\ 0 & 0 & 1 & 1 & 8 & 3 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right) \begin{array}{l} -\frac{1}{3} R_1 \to R_1 \\ R_2 - R_3 \to R_2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 6 & 1 & 0 & 7 & -1 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right) \quad R_1 - R_2 \to R_1$$

$$\begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right) \\ \begin{array}{ccccc} x_1 & r & x_3 & x_4 & s \end{array} \end{array} \quad \text{Reduced form}$$

Solution: $x_1 = -2 - 6r - 4s$
$x_2 = r$
$x_3 = 1 - 3s$
$x_4 = 2 - 5s$
$x_5 = s$

$r, s \in \mathbb{R}$ are parameters.

(b) [8 marks]

$$\left(\begin{array}{cc|c} 2 & 3 & 2 \\ 1 & 4 & 6 \\ 5 & k & 2 \end{array}\right) \; R_1 \leftrightarrow R_2 \; \left(\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 3 & 2 \\ 5 & k & 2 \end{array}\right) \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 5R_1 \to R_3 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & -5 & -10 \\ 0 & k-20 & -28 \end{array}\right) \; -\frac{1}{5} R_2 \to R_2 \; \left(\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 1 & 2 \\ 0 & k-20 & -28 \end{array}\right) \; R_3 - (k-20)R_2 \to R_3$$

$-28 - 2k + 40$

$$\left(\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & -2k+12 \end{array}\right)$$

We have a solution iff $-2k + 12 = 0$ i.e. $k = 6$. Then sol^{n} is $x = -2$, $y = 2$.

[15marks]

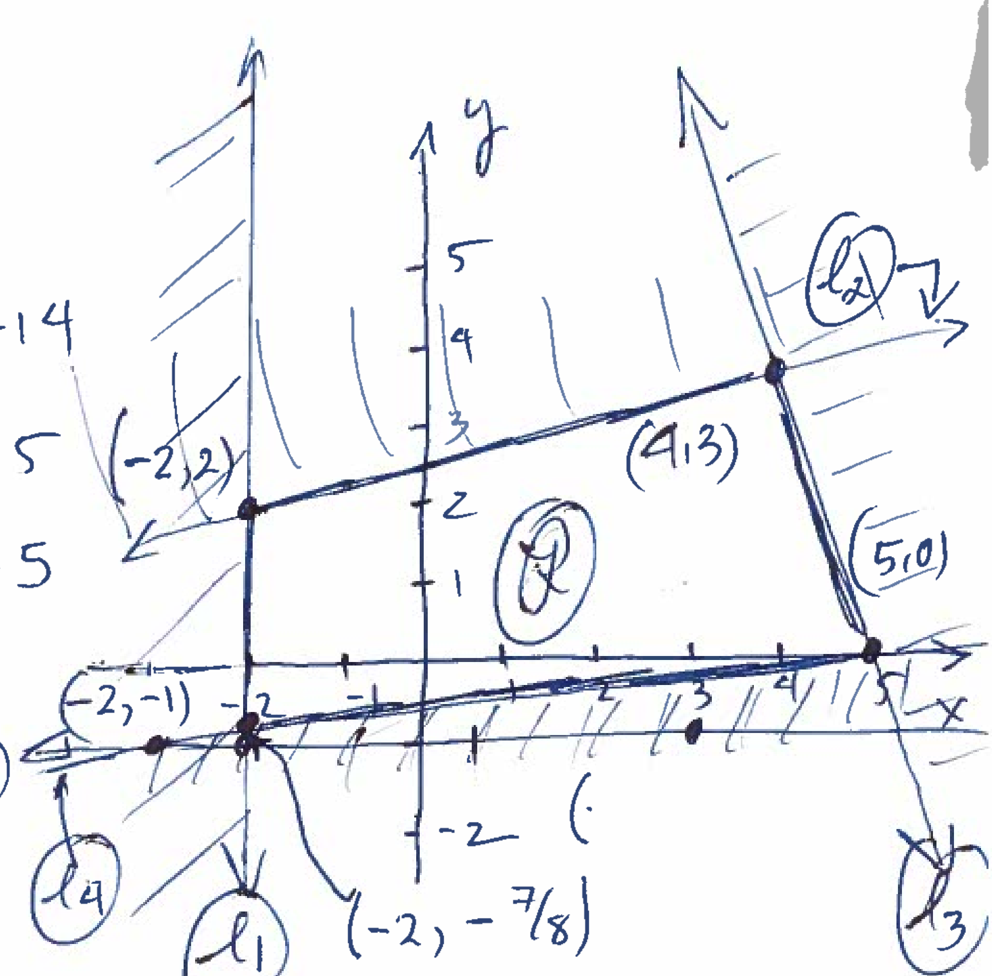
3. l_1 is $x=-2$

l_2 is $x-6y=-14$

l_3 is $3x+y=15$

l_4 is $x-8y=5$

l_5 is $y=-1$



$\underline{\underline{l_2}}$: $6y = x+14 \Rightarrow y = \frac{x}{6} + \frac{14}{6}$

$\underline{l_3}$: $y = -3x+15$

$\underline{l_2 \cap l_3}$:

$\frac{x}{6} + \frac{14}{6} = -3x+15$

$\frac{19}{6}x = \frac{76}{6} \Rightarrow x=4,\ y=3$

$\underline{\underline{l_4}}$: $x - 8y = 5 \Rightarrow 8y = x-5$

$y = \frac{1}{8}x - \frac{5}{8}$

$\underline{l_3 \cap l_4}$:

$x - 8(-3x+15) = 5$

$25x = 125$

$\Rightarrow x = 5 \quad y=0$

When $x=-2$, $-2-8y=5$

Sub-in l_4 $\quad -8y = 7 \quad y = -\frac{7}{8}$ Point $= (-2, -7/8)$

$\underline{\text{Corner pts}}$ of feasible set are: $\quad R$ = feasible set

$(-2,2),\ (4,3),\ (5,0),\ (-2,-\frac{7}{8})$ $\quad$ R is non-empty, bounded, standard

$\underline{\text{FTLP}} \Rightarrow$ We use corner pt eval:

$Z(-2,2) = 6(-2)+2(2) = -12+4 = -8$

$Z(4,3) = 6(4)+2(3) = 24+6 = 30$

$Z(5,0) = 6(5)+0 = 30$

$Z(-2,-\frac{7}{8}) = 6(-2)+2(-\frac{7}{8}) = -12 - \frac{14}{8} = \frac{-110}{8} = -\frac{55}{4}$ = MIN

MAX = 30 on segment (4,3) to (5,0)

(4) (a) $A - xI = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & 5 & 1 \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$

[7 marks]

$$= \begin{pmatrix} -x & 0 & 0 \\ 0 & -6-x & -2 \\ 0 & 5 & 1-x \end{pmatrix}$$

$$\det(A - xI) = (-x)(-1)^{1+1} \det\begin{pmatrix} -6-x & -2 \\ 5 & 1-x \end{pmatrix}$$

$$= -x\left[(-6-x)(1-x) + 10\right]$$

$$= (-x)\left[-6+6x-x+x^2+10\right]$$

$$= (-x)\left[x^2+5x+4\right] = (-x)(x+1)(x+4)$$

$\det(A - xI) \neq 0 \iff A - Ix$ invertible

$\iff x \in \mathbb{R},\ x \notin \{0, -1, -4\}$.

(or $x \neq 0, -1, -4$)

(b) $A - I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -7 & -2 \\ 0 & 5 & 0 \end{pmatrix}$

[8 marks]

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -7 & -2 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -R_1 \to R_1 \\ R_2 \leftrightarrow R_3 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 1 \\ 0 & -7 & -2 & 0 & 1 & 0 \end{array}\right) \tfrac{1}{5}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/5 \\ 0 & -7 & -2 & 0 & 1 & 0 \end{array}\right) R_3 + 7R_2 \to R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & -2 & 0 & 1 & 7/5 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1 & 0 & -1/2 & -7/10 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} \\ 0 & -\frac{1}{2} & -\frac{7}{10} \end{pmatrix}$$

5 (a) x = amount in 5%
y = amount in 7%
z = amount in 8%

[4 marks]

We have
$$x + y + z = 1,600$$
$$3x - z = 0$$
$$.05x + .07y + .08z = 115$$

Use equations
$$\boxed{\begin{aligned} x + y + z &= 1600 \\ 3x - z &= 0 \\ 5x + 7y + 8z &= 11500 \end{aligned}}$$

(b)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 3 & 0 & -1 & 0 \\ 5 & 7 & 8 & 11500 \end{array}\right) \begin{array}{l} R_2 - 3R_1 \to R_1 \\ R_3 - 5R_1 \to R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & -3 & -4 & -4800 \\ 0 & 2 & 3 & 3500 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & 1 & \frac{4}{3} & 1600 \\ 0 & 1 & \frac{3}{2} & 1750 \end{array}\right) R_3 - R_2 \to R_3 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & 1 & 4/3 & 1600 \\ 0 & 0 & \frac{1}{6} & 150 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1600 \\ 0 & 1 & 4/3 & 1600 \\ 0 & 0 & 1 & 900 \end{array}\right) \begin{array}{l} R_1 - R_3 \to R_1 \\ R_2 - \frac{4}{3}R_3 \to R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 700 \\ 0 & 1 & 0 & 400 \\ 0 & 0 & 1 & 900 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 400 \\ 0 & 0 & 1 & 900 \end{array}\right)$$

[8 marks]

$\therefore x = 300, \; y = 400, \; z = 900.$

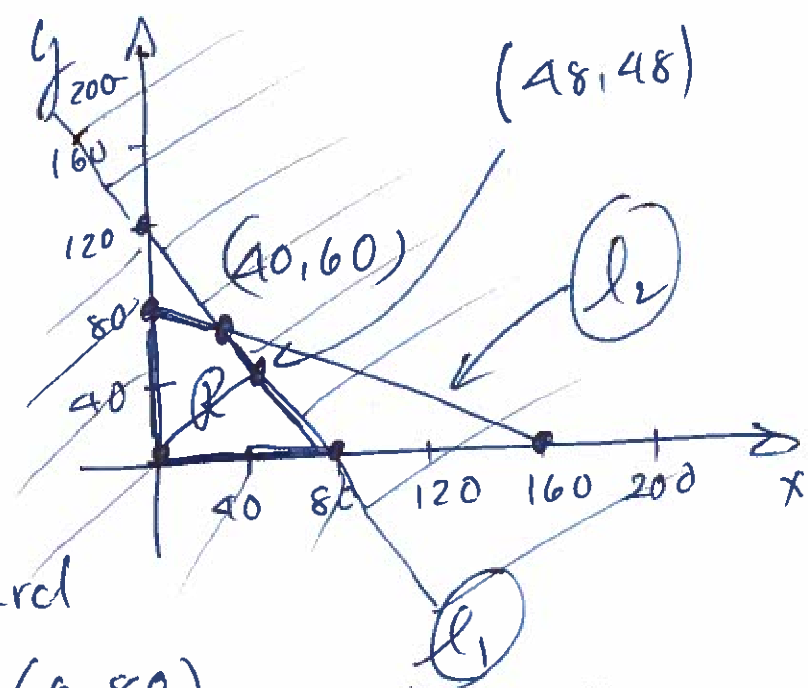
⑥ x = number of A made
y = number of B made,

(a) Profit = Revenue − Cost [10 marks]

$$= \underbrace{500x + 400y}_{\text{objective } f^n} = Z$$

Constraints: $3x + 2y \le 240$
$\frac{1}{2}x + y \le 80$
$x \geq 0 \quad y \geq 0$

$l_1: 3x + 2y = 240$
$l_2: x + 2y = 160$
$l_1 \cap l_2: 2x = 80$
$x = 40, \; y = 60$



R is bounded, ≠ φ, standard
Corner pts are (0,0), (0,80), (40,160), (80,0)

FTLP ⟹ Corner Pt eval:
$Z(0,0) = 0$
$Z(80,0) = 40{,}000$
$Z(40,160) = 20{,}000 + 24\,000 = 44{,}000$
$Z(0,80) = 32{,}000$

MAX @ $x = 40$, $y = 60$ of \$44,000

(b) When $x = y$ [4 marks]
$Z(48,48) = (900)(48) = 43{,}200$
@ $48 = x = y$

⑦ A is $n \times n$ $\quad R_i = i^{th}$ row of A

Assume $1R_1 + 2R_2 + 3R_3 + \cdots + nR_n = 0$.

i^{th} column of $A^T = i^{th}$ row of A [5marks]

$$A^T = \begin{pmatrix} | & | & | & & | \\ R_1 & R_2 & R_3 & \cdots & R_n \\ | & | & | & & | \end{pmatrix}$$

"Assumption" $\Rightarrow A^T \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\therefore$ homogeneous system $A^T X = 0$ has a non-trivial sol$^{\underline{n}}$, namely $\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$

$\therefore A^T$ is not invertible

$\therefore A$ is not invertible. $\blacksquare$

BIG END