# ***** Sorry...No solutions will be provided *****

## University of Toronto at Scarborough
## Department of Computer and Mathematical Sciences

## FINAL EXAMINATION

## MATA33 - Calculus for Management II

Examiners: R. Grinnell
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Date: April 24, 2012
Time: 9:00 am
Duration: 3 hours

## Provide the following information:

**Last Name (PRINT):** ______________________________

**Given Name(s) (PRINT):** ______________________________

**Student Number (PRINT):** ______________________________

**Signature:** ______________________________

## Read these instructions:

1. This examination has 14 numbered pages. It is your responsibility to ensure that all of these pages are included.
2. Put your solutions and/or rough work in the answer spaces provided. If you need extra space, use the back of a page or blank page 14. Clearly indicate your continuing work.
3. Show all work and justify your answers. Full points are awarded for solutions that are correct, complete, and show relevant concepts from MATA33.
4. You may use one standard hand-held calculator (a graphing facility is permitted). All other electronic devices (e.g. cell phone, smart phone, i-Pod), extra paper, notes, textbooks, and backpacks are forbidden at your workspace either by intent or accident.
5. You may write in pencil, pen, or other ink.

## Print letters for the Multiple Choice Questions in these boxes:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

## Do not write anything in the boxes below.

| **A** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **TOTAL** |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | |
| 40 | 12 | 15 | 7 | 7 | 16 | 14 | 14 | 16 | 9 | 150 |

**Part A: 10 Multiple Choice Questions Print the letter of the answer you think is most correct in the boxes on the first page.** Each right answer earns 4 points and no answer/wrong answers earn 0 points.

1. Let $A = \begin{bmatrix} 9 & 5 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 6 & 12 \end{bmatrix}$ The matrix $6\,A^{-1} - B^T$ is

   (A) $\begin{bmatrix} 7 & -16 \\ -12 & 30 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 16 \\ 0 & 6 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -16 \\ 0 & -6 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & -16 \\ 0 & 6 \end{bmatrix}$

2. Given are $2 \times 2$ matrices $A = [A_{ij}] = [i + 2j]$ and $B$, where $det(B) = -3$ . The value of $det\left(3A^2B^{-1}\right)$ is

   (A) $-4$ (B) $8/3$ (C) $-12$ (D) $12$ (E) none of (A) - (D)

3. If $f(x, y) = x^3\,ln(y) + 2\sqrt{xy}$ then $f_x(1, e)$ equals

   (A) $3 + \left(\frac{\sqrt{e}}{2}\right)$ (B) $3 + 2\sqrt{e}$ (C) $2\sqrt{e}$ (D) $3 - \sqrt{e}$ (E) $3 + \sqrt{e}$

4. Let $Z = -x - y$. If $R$ is the feasible region defined by $x \geq 0$ and $-x \leq y \leq 0$ , then which one of the following statements is true?

(A) $Z$ has both a maximum and a minimum on $R$.

(B) $Z$ has a maximum, but not a minimum on $R$.

(C) $Z$ has a minimum, but not a maximum on $R$.

(D) $Z$ has neither a maximum nor a minimum on $R$.

(E) The optimization of $Z$ on $R$ is uncertain because the feasible region is unbounded.

5. The joint demand functions for two products $A$ and $B$ are given by $a(x, y) = 150 - 3x - e^{-2y}$ and $b(x, y) = -20y + \dfrac{50y}{(x+2)}$ respectively, where $x > 0$ and $y > 0$ are the unit prices of $A$ and $B$, respectively. We may conclude that the products are

(A) competitive (B) complementary (C) neither (A) nor (B) (D) both (A) and (B)

6. If the equation $z^3 + z + 3 = 4x - xy$ defines $z$ implicitly as a function of independent variables $x$ and $y$, then the value of the mixed partial derivative $z_{xy}$ when $x = 5$ and $z = 1$ is

(A) $-3/8$ (B) $-1/4$ (C) $-1/2$ (D) $1/4$ (E) a number not in (A) - (D)
(F) unknown because we do not have a value for $y$.

7. The value of $\int_0^4 \int_1^2 2x\sqrt{y}\ dx\ dy$ is

(A) $16$ (B) $12$ (C) $8\sqrt{2}$ (D) $\sqrt{10}$ (E) none of (A) - (D)

8. If $z = x^2 + 3xy^4$ and $x = \left(e^{r-1}\right)^3 + 2s$ and $y = \dfrac{r}{s+1}$ then the value of $\dfrac{\partial z}{\partial s}$ at $r = 1$ and $s = 0$ is

(A) $0$ (B) $-12$ (C) $-7$ (D) $-2$ (E) $22$ (F) none of (A) - (E)

9. Exactly how many of the following properties are mathematically equivalent to the statement,

"The $3 \times 3$ matrix $P$ is invertible"?

- $QP = I$ for some $3 \times 3$ matrix $Q$ ($I$ is the $3 \times 3$ identity matrix).
- The reduced form of $P$ is $I$.
- $det(P) \neq 0$
- The $3 \times 1$ zero matrix is a solution to the matrix equation $PX = 0$ ($X$ is a $3 \times 1$ matrix of variables and 0 is the $3 \times 1$ zero matrix)

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

10. Exactly how many of the following mathematical statements are always true?

- If $P(x, y)$ is a polynomial function in the variables $x$ and $y$, then the second-order partial derivatives of $P$ are equal.
- A function $h(x, y)$ has a relative maximum at a point $(a, b)$ if and only if $(a, b)$ is a critical point of $h$ and $h_{xx}(a, b) < 0$.
- If a function $f(x, y)$ has a relative minimum at a point $(a, b)$ and $(a, b)$ satisfies a constraint $g(x, y) = c$, then the point $(a, b, \lambda)$ is a critical point of the Lagrangian of $f$ and $g$.
- If $Q(x, y)$ is a polynomial function in the variables $x$ and $y$ then $\int_0^1 \int_0^x Q(x, y)\, dy\, dx = \int_0^1 \int_0^y Q(x, y)\, dx\, dy$ .

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

**BE SURE YOU HAVE PUT YOUR ANSWERS IN THE 1-ST PAGE BOXES**

**Part B: 9 Full Solution Questions** Full points are awarded only if your solutions are correct, complete, and sufficiently display relevant concepts from MATA33.

1. A company is developing a new soft drink. The cost in dollars to produce a 100 litre batch is approximately $C(x, y) = 27x^3 - 72xy + 8y^2 + 2,200$ where $x \geq 0$ is the number of kilograms of sugar used and $y \geq 0$ is the number of kilograms of flavouring used. Find the amounts of sugar and flavouring that minimize the production cost per 100 litre batch. What is the minimum cost? [12 points]

2. In all of this question let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 3 & 2 \end{pmatrix}$ and for each real number $x$, let $B(x)$ be the $3 \times 3$ matrix $xI - A$ where $I$ is the $3 \times 3$ identity matrix.

(a) Find all values of $x$ for which the matrix $B(x)$ is invertible. [8 points]

(b) Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let $C = B(3)$. <u>Use Cramer's rule</u> to solve for the variable $x_2$ in the matrix equation $CX = \begin{pmatrix} 0 \\ -8 \\ -8 \end{pmatrix}$. [7 points]

3. Let $g(x, y) = \sqrt{y - x^2 + 2} \; + \; ln(2 - y)$. State the domain of $g$ using "set-bracket notation" (i.e. starting like $\{(x, y) \ldots\}$) and appropriate inequalities, then make a clear, accurate sketch of the domain.

[7 points]

4. The "Cobb-Douglas" production function has the form $P = M s^{\alpha} t^{\beta}$ where $s, t > 0$ are variables; $M, \alpha, \beta$ are positive constants; and $\alpha + \beta = 1$.

Show that $s^2 P_{ss} + t^2 P_{tt} = (-2\alpha\beta) P$. [7 points]

5. Assume the function $f(x, y, z) = 3x + 2y + z$ has absolute extrema subject to the two constraints $x + y + z = 1$ and $x^2 + y^2 = 5$. Use the method of Lagrange Multipliers for two constraints to find the points where the absolute extrema occur and find the absolute extreme values. [16 points]

6. A machine shop makes two types of bolts: Type A and Type B. The manufacturing of each type requires time (in minutes) on three machines: Machine 1, Machine 2, and Machine 3. The time required for each bolt type on each machine is given in the table below:

| | Type A | Type B |
|---|---|---|
| Machine 1 | 0.2 min | 0.2 min |
| Machine 2 | 0.6 min | 0.2 min |
| Machine 3 | 0.04 min | 0.08 min |

On a typical day of production at most 300, 720, and 100 minutes are available on Machines 1, 2, and 3, respectively. Type A bolts sell for 15 cents and Type B bolts sell for 20 cents. How many of each bolt type should be manufactured per day to maximize revenue? A complete solution requires a statement of a set of linear inequalities, an accurate sketch of the feasible region, and appropriate details. [14 points]

7. (a) Let $u(x,y) = e^{x^2 y}$ . Find the function $m(x,y)$ such that $u_{yy} + u_{yx} = u(x,y)\, m(x,y)$

[8 points]

(b) Let $a, w > 0$ be constants. Assume a function $F = aP - whL$ where $P = f(x,y)$ for some function $f$; $x = Lg(h)$ for some function $g$; and $y$ is a variable independent of variables $L$ and $h$ (and $L$ and $h$ are independent of each other).

Prove that $\dfrac{\partial F}{\partial L} = a \dfrac{\partial P}{\partial x} g(h) - wh$ [6 points]

8. Find and classify all critical points of the function $f(x,y,z) = yz^2 - 3z^2 + y^3 + 3y^2 - x^2 + 8x$

[16 points]

9. In all of this question let $I = \int_0^4 \int_{y/4}^1 e^{x^2}\, dx\, dy$ [9 points]

(a) Accurately sketch the region $\mathcal{R}$ over which the integration in $I$ takes place.

(b) Find the value of $I$.

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