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University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

FINAL EXAMINATION

MATA33 - Calculus for Management II

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Date: April 20, 2011
Time: 7:00 pm
Duration: 3 hours

Provide the following information:

Last Name (PRINT): ____________________

Given Name(s) (PRINT): ____________________

Student Number (PRINT): ____________________

Signature: ____________________

Read these instructions:

1. This examination booklet has 14 numbered pages. It is your responsibility to ensure that at the beginning of the exam all of these pages are included.
2. Put your solutions and/or rough work in the answer spaces provided. If you need extra space, use the back of a page or the blank page at the end of the exam booklet.
3. You may use one standard hand-held calculator (a graphing facility is permitted). All other electronic devices (e.g. cell phone, smart phone, i-Pod), extra paper, notes, textbooks, and backpacks are forbidden at your workspace.
4. If you brought a cell phone into the exam room, it must be turned off and left at the front of the exam room.

Print letters for the Multiple Choice Questions in these boxes:

1	2	3	4	5	6	7	8	9

Do not write anything in the boxes below.

A	1	2	3	4	5	6	7	8	9	**TOTAL**
36	11	14	12	12	14	11	14	12	14	150

Part A - Multiple Choice Questions For each of the following **print the letter of the answer you think is most correct in the boxes on the first page**. Each right answer earns 4 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. Let $A = \begin{bmatrix} 5 & 8 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 0 & 6 \end{bmatrix}$. The matrix $A^{-1} + B^T$ is

   (A) $\begin{bmatrix} 7 & -8 \\ 1 & 11 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 8 \\ 5 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 8 \\ 5 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 8 \\ 5 & 1 \end{bmatrix}$ (E) none of (A) - (D)

2. If $P = \begin{bmatrix} 4 & 8 \\ 1 & 3 \end{bmatrix}$ and Q is a 2×2 matrix such that $det(Q) = 2$ then $det(\frac{1}{2}P^2Q^{-1})$ equals

   (A) 4 (B) 9/4 (C) 20 (D) 2 (E) 9/8 (F) a number not in (A) - (E)

3. Exactly how many of the following properties are mathematically equivalent to the following statement: "A 3×3 matrix K is invertible"?

   - $KH = HK$ for some 3×3 matrix H.
   - $det(K) \neq 0$.
   - The reduced form of K is the 3×3 identity matrix.
   - The matrix equation $KX = 0$ has the trivial solution (0 is the 3×1 zero matrix).
   - K has neither a row of zeros nor a column of zeros.

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) 5

4. A production function is $P(l,k) = l^{0.6}k^{0.4}$ where l is a measure of labour used and k is a measure of capital used. The marginal productivity with respect to labour when $l = 20$ and $k = 6$ is a number

(A) < 0.2 (B) in $[0.2,\ 0.3)$ (C) in $[0.3,\ 0.4)$ (D) in $[0.4,\ 0.5)$ (E) ≥ 0.5

5. The joint demand functions for two products X and Y are given by $a(x,y) = \dfrac{100}{x\sqrt{y}}$ and $b(x,y) = \dfrac{300}{y\sqrt[5]{x}}$ respectively, where x and y are the unit prices of X and Y, respectively. We may conclude that the products are

(A) competitive (B) complementary (C) neither (A) nor (B) (D) both (A) and (B)

6. Let $c(x,y)$ be a cost function for the products X and Y as in Question 5 above. Assume $c(20,\ 13) = 853$, $c_x(20,\ 13) = -2$, $c_x(21,12) = 1$, and $c_y(20,12) = -3$. The value of $c(20,12)$ is approximately

(A) 856 (B) 855 (C) 851 (D) 850 (E) a number not in (A) - (D)

7. Assume a function $w = f(x, y, z)$ has continuous second-order partial derivatives at all points (x, y, z) and that (a, b, c) is a critical point of f where $a < c < 0$ and $b > 1$. Assume the Hessian matrix (i.e. the matrix of second partial derivatives) is $A = Hf(a, b, c) = \begin{bmatrix} ab & a & 0 \\ a & a & 0 \\ 0 & 0 & c \end{bmatrix}$. We may conclude that

(A) f has a relative minimum at (a, b, c)

(B) f has a relative maximum at (a, b, c)

(C) f has no relative extrema at (a, b, c)

(D) the second derivative is inconclusive

(E) none of (A) - (D) are correct

8. The value of $\int_0^1 \int_1^3 xy^2 \, dy \, dx$ is

(A) 13/3 (B) 5/3 (C) 4/3 (D) 13/2 (E) 13/9 (F) none of (A) - (E)

9. Exactly how many of the following mathematical statements are always true?

- A function $f(x, y)$ has a relative extrema at a point (a, b) if and only if (a, b) is a critical point of f.

- If $g(x, y)$ is continuous on a region R in the plane, then $\int\int_R g(x, y) \, dA$ represents a volume.

- If $f(x, y)$ has relative minimum at a point (a, b) and (a, b) satisfies a constraint $g(x, y) = c$, then (a, b, λ) is a critical point of the Lagrangian for f and g.

- If $h(x, y)$ is a function for which all second-order partial derivatives are continuous at every point (x, y), then the second-order partial derivatives are equal.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

(Check that you have printed the correct letter answers in first page boxes)

Part B - Full Solution Questions Full points are awarded only if your solutions are correct, complete, and sufficiently display relevant concepts from MATA33.

1. Let $f(x, y) = 2x^2 + y^2 - x^2y$. Find the critical point(s) of f. For each one, use the second derivative test to determine whether it corresponds to a relative maximum, minimum, or a saddle point. [11 points]

2. (a) Let $u = \dfrac{3x^2+4}{y+1}$ where $x = re^{7s}$ and $y = 1 + r(s+2)^2$. Use the chain rule for functions of two variables to calculate the value of $\dfrac{\partial u}{\partial s}$ when $r = 2$ and $s = 0$.

[8 points]

(b) Assume the equation $z^3 + x^2 z = 4x + yz$ defines z implicitly as a function of independent variables x and y. Find z_x at the point where $y = -3$ and $z = 1$.

[6 points]

3. In all of this question let $\mathcal{R}$ represent the feasible region in the plane whose corner points are $A = (-1,-1)$, $B = (0,3)$, $C = (2,3)$, $D = (5,1)$, and $E = (0,-2)$.

   Let $Z = F(x,y) = 2rx + 3ry$ be an objective function where $r < 0$ is an arbitrary constant.

   (a) Maximize and minimize the objective function F on $\mathcal{R}$. State the maximum and minimum values and state all points where these values occur. Include a sketch of $\mathcal{R}$.

   [8 points]

   (b) Let $\mathcal{S}$ be the feasible region obtained from $\mathcal{R}$ by removing the line segment DE from $\mathcal{R}$. Show that the objective function F has no maximum value on $\mathcal{S}$. [4 points]

4. A company sells two products, A and B. The revenue function (in dollars per week) is $R(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y$ where x and y are the number of units of A and B sold per week, respectively. The associated cost function is $C(x, y) = 180x + 140y + 5000$ dollars per week. Find the value(s) of x and y that maximize the weekly profit. Do verify that your findings actually give maximum profit. (Recall that Profit = Revenue - Cost).

[12 points]

5. Imagine a few years after graduation from UTSC that you are employed as the senior buyer for a specialty European car dealership in Toronto. Your dealership sells two models of a certain luxury car: the sport model (which costs the dealership $\$c$ to buy) and the touring sedan (which costs the dealership $\$d$ to buy). It is the case that $d > c$. You have a dealership budget of $\$B$ to buy a sum total of k of these cars. Assume all of the budget is used and that you buy $x > 0$ sport models and $y > 0$ touring sedans.

   (a) State the system of two linear equations in the letters $x,\ y,\ c,\ d,\ k, and\ B$ that describe the "buying situation" outlined above. [2 points]

   (b) Use Cramer's rule to solve for x and y in terms of the other four constants. [6 points]

   (c) Use your answer to part (b) to show that $c < \dfrac{B}{k} < d$. [3 points]

   (d) Assuming the inequality in part (c) is true, give a short reason why the values of x and y in part (b) may not actually be meaningful in the context of the "buying situation" outlined above. [3 points]

6. In all of this question let $f(x,y) = \dfrac{\sqrt{y - x^2 + 4}}{\sqrt{5 - y}}$

(a) Let D represent the domain of f. State D using set-notation and appropriate inequalities and then give a good sketch that clearly shows D. [6 points]

(b) Find the function $y = g(x)$ that gives the level curve of f passing through the point $(2, 4)$. [5 points]

7. In all of this question let $C(x,y) = \dfrac{xy}{2x+5y}$ be the total manufacturing cost (in millions of dollars) for $x > 0$ hundreds of units of product A and $y > 0$ hundreds of units of product B.

(a) Find and simplify the marginal cost functions $C_x(x,y)$ and $C_y(x,y)$. [8 points]

(b) Find the number of units of products A and B manufactured under the following two assumptions: (i) the sum total of units manufactured is $2,000$ and (ii) the marginal cost functions are equal. Round your answers down the nearest unit. [6 points]

8. The Cobb-Douglas production function for a company is given by $P = f(x, y) = 500x^{2/5}y^{3/5}$ where $x > 0$ is the units of labour (at \$100 per unit) and $y > 0$ is the units of capital (at \$250 per unit). The total budget for labour and capital is \$100 thousand and this is all used. Use the method of Lagrange multipliers to find x and y that yields the maximum production and state this maximum value (You may assume that such a maximum actually exists).

[12 points]

9. (a) Find $\displaystyle\int\int_R x + y \; dA$ where R is the triangular region with vertices $(0,0)$, $(2,2)$, and $(4,0)$. [7 points]

(b) Evaluate $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \sqrt{4 + y^3} \; dy \; dx$. (Hint: consider reversing the order of integration).

[7 points]

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