# *** Sorry...no solutions will be provided ***

## University of Toronto at Scarborough
## Department of Computer and Mathematical Sciences

## FINAL EXAMINATION

## MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: August 13, 2011
Time: 7:00 pm
Duration: 3 hours

## Provide the following information:

**Last Name (PRINT):** ____________________

**Given Name(s) (PRINT):** ____________________

**Student Number (PRINT):** ____________________

**Signature:** ____________________

## Read these instructions:

1. This examination has 14 numbered pages. It is your responsibility to ensure that all of these pages are included.
2. Put your solutions and/or rough work in the answer spaces provided. If you need extra space, use the back of a page or blank page 14. Clearly indicate your continuing work.
3. Show all work and justify your answers. Full points are awarded for solutions that are correct, complete, and show relevant concepts from MATA33.
4. You may use one standard hand-held calculator (a graphing facility is permitted). All other electronic devices (e.g. cell phone, smart phone, i-Pod), extra paper, notes, textbooks, and backpacks are forbidden at your workspace either by intent or accident.
5. If you have a cell phone in your possession, it must be turned off and left at the front of the exam room.
6. You may write in pencil, pen, or other ink.

## Print letters for the Multiple Choice Questions in these boxes:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

## Do not write anything in the boxes below.

| **A** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | **TOTAL** |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
| 30 | 16 | 16 | 16 | 13 | 18 | 12 | 16 | 13 | 150 |

**Part A: 10 Multiple Choice Questions** For each of the following **print the letter of the answer you think is most correct in the boxes on the first page**. Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. If $w = x^3 e^{xy}$ and $x \neq 0$ then $\dfrac{\partial w}{\partial x}(x, y)$ equals

   (A) $\dfrac{w}{x}(3+x)$ (B) $\dfrac{w}{y}(3+xy)$ (C) $\dfrac{w}{x}(3+xy)$ (D) $w(3+xy)$ (E) $\dfrac{w}{y}(3+x)$

2. Let $C = \begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 2 \\ 11 & 3 \end{bmatrix}$. The matrix $4C^T B^{-1}$ is

   (A) $\begin{bmatrix} -54 & 40 \\ 20 & -14 \end{bmatrix}$ (B) $\begin{bmatrix} -54 & 40 \\ -20 & -12 \end{bmatrix}$ (C) $\begin{bmatrix} -54 & 40 \\ 20 & -12 \end{bmatrix}$ (D) $\begin{bmatrix} 54 & 40 \\ 20 & -12 \end{bmatrix}$

   (E) none of (A) - (D)

3. The equation of the level curve of $f(x, y) = \dfrac{x}{y^2 + 3} - 4$ that passes through $(4, 1)$ is

(A) $4y^2 = x$ (B) $x = y^2 + 3$ (C) $2y^2 + 2 = x$ (D) $4x = 11y^2 + 5$

(E) none of (A) - (D)

4. Let $a > b > 0$ and $Z = ax - by$. If $R$ is the feasible region defined by $y \geq x \geq 0$ and $y \geq 1$, then which one of the following statements is true?

(A) $Z$ has both a maximum and a minimum on $R$.

(B) $Z$ has a maximum, but not a minimum on $R$.

(C) $Z$ has a minimum, but not a maximum on $R$.

(D) $Z$ has neither a maximum nor a minimum on $R$.

(E) The optimization of $Z$ on $R$ is uncertain because the exact values of $a$ and $b$ are unknown.

5. Exactly how many of the following statements are always true?

- Every system of $m \geq 2$ linear equations in $n > m$ variables has infinitely many solutions.
- If $A$ is a $3 \times 3$ matrix and $AX = 0$ has the trivial solution, then $A$ is invertible.
- If $P$ and $Q$ are non-zero $2 \times 2$ matrices such that $P^2 = Q^2$, then either $P = Q$ or $P = -Q$.
- If $B$ and $C$ are invertible matrices, then they have the same reduced form.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

6. The joint demand functions for two products $A$ and $B$ are given by $a(x,y) = (x+1)e^{-x+3y}$ and $b(x,y) = 4x^{1/2}y^{-1/3} + ln(y+2)$ respectively, where $x > 0$ and $y > 0$ are the unit prices of $A$ and $B$, respectively. We may conclude that the products are

(A) competitive (B) complementary (C) neither (A) nor (B) (D) both (A) and (B)

7. If $z = \dfrac{x^2y}{y+1}$ then the value of the mixed partial derivative $z_{xy}(2,1)$ equals

(A) 2 (B) 4 (C) 1 (D) $-1$ (E) 0 (F) none of (A) - (E)

8. Assume the equation $uy^2 = e^u + x$ defines $u$ implicitly as a function of independent variables $x$ and $y$. Then the partial derivative $u_x$ is equal to

(A) $1 + e^u - y^2$ (B) $y^2 - e^u$ (C) $(e^u + 1)y^{-2}$ (D) $(y^2 - e^u)^{-1}$ (E) none of (A) - (D)

9. At the point $(1,1)$ the function $f(x,y) = x^2 - 2x + y^2$ has

(A) a local minimum (B) a local maximum

(C) a saddle point (D) none of (A) - (C)

10. The value of $\int_0^2 \int_1^3 y^3 \, dx \, dy$ is

(A) 32 (B) 13/4 (C) 4 (D) 9/2 (E) 8 (F) none of (A) - (E)

**(Check that you have printed the correct letter answers in first page boxes)**

**Part B: 8 Full Solution Questions** Full points are awarded only if your solutions are correct, complete, and sufficiently display relevant concepts from MATA33.

1. (a) Let $w = f(x,y) = 5x^2\sqrt{y+1}$ where $x = 2r^3 + 4s^2$ and $y = r(s+7)^{2/3}$.

   Find $\dfrac{\partial w}{\partial s}$ when $r = 2$ and $s = 1$. Give your answer as a simplified rational number, not a decimal.

   [8 points]

   (b) Assume $z$ is defined implicitly as a function of independent variables $x$ and $y$ by the equation $z^3 + 2x^2z^2 = 5xy$. Calculate the value of $y$ when $x = z = 2$, and then find the partial derivative $z_x$ evaluated at the point $(2, y, 2)$. Give your answers as simplified rational numbers, not decimals. [8 points]

2. (a) Let $g(x, y) = \sqrt{-2x^2 + 2 - y} + ln(y)$. State the domain of $g$ using "set-bracket notation" (i.e. starting like $\{(x, y) \ldots\}$) and appropriate inequalities, then make a clear, accurate sketch of the domain.

[8 points]

(b) Find the equation of the horizontal plane that is tangent to the graph of the function $z = H(x, y) = 3x^2 - 6xy + 2y^2 + 30x - 12y + 27$ and find the point of tangency.

[8 points]

3. Let $V$ be a positive, real constant and consider all possible rectangular boxes that have volume $V$ and no top (thus each box has five faces: the bottom, front and back face, and the left and right face). Calculate the dimensions of such a box that requires the least amount of material for its construction. You may assume no material is wasted. Show all work and appropriate justifications. [16 points]

4. The production function for a company is given by $P = f(x, y) = 3x^2 + xy + 2y^2$ where $x > 0$ is the units of labour (at \$5 per unit) and $y > 0$ is the units of capital (at \$2 per unit). The total budget for labour and capital is \$5,200 and this is all used. Use the method of Lagrange Multipliers to find the value of $x$ and $y$ that yield the maximum production subject to the total budget constraint above. You may assume: (i) the critical point obtained actually does correspond to maximum production and (ii) fractional amounts of units are possible.

[13 points]

5. (a) Assume $A$ and $B$ are $4 \times 4$ matrices such that $det(A) = -5$ and $det(B) = 2$.

(i) $det\left(3AB^2\right) =$ [3 points]

(ii) $det\left(det(B^{-1})A\right) =$ [2 points]

(iii) Prove there does not exist a $4 \times 4$ matrix $C$ (with real entries) such that $C^2 = A$. [3 points]

(b) For each real number $t$, let $S(t)$ represent the system of linear equations in variables $x$ and $y$:

$$2tx + 3y = 1$$
$$(t + 5/3)x + ty = 4$$

Find all values of $t$ for which $S(t)$ has a unique solution. Then use Cramer's rule to solve for $x$ and $y$ in terms of these values $t$. [10 points]

6. In all of this question let $E = g(r,x) = \left(1+\frac{r}{x}\right)^x - 1$ represent the "effective rate" function (where $r =$ the A.P.R. and $x =$ the number of annual compounding periods. You may assume both $r$ and $x$ are positive real variables).

(a) Verify that $(x+r)E_r = x(E+1)$ (Remember that $E_r = \dfrac{\partial E}{\partial r}$ ) [5 points]

(b) Prove that $\dfrac{\partial E}{\partial x} = (E+1)\left[\, ln\left(1+\frac{r}{x}\right) - \dfrac{r}{x+r} \,\right]$ [7 points]

7. Find and classify all critical points of the function $f(x, y, z) = xy^2 - 3y^2 + x^3 + 3x^2 - z^2 + 8z$

[16 points]

8. In all of this question let $R = \{(x, y) \mid x \geq 0, \;\; y \leq 3, \;\; 3x \leq 4y\}$.

(a) Draw $R$ and find $\displaystyle\iint_R 4 - 2y \; dA$. [2, 6 points]

(b) Find the value of the constants $c$ and $d$ such that $c < 0 < d$ and the function $F(x, y) = cx + dy$ has maximum value of 33 and minimum value of $-9$ on $R$.

[5 points]

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