SOLUTIONS

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test # 2 — See * below

MATA32 - Calculus for Management I

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Date: November 8, 2008
Duration: 100 minutes

Clearly indicate the following information:

Surname: SOLUTIONS

Given Name(s): ____________________

Student Number: ____________________

Signature: ____________________

Tutorial Number (e.g. TUT0032): ____________________

Carefully circle the name of your Teaching Assistant:

Talmage ADAMS
Eric CORLETT
Minh DANG
Mikhail GUDIM
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Alex LUCAS
Chris LUI
Molu SHI
Sean TRIM
Alfred YIP
Fan ZHANG
Yichao ZHANG

Read these instructions:

1. This midterm test has 10 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space use the back of a page and clearly indicate the location of your continuing work.
3. For the Part B Questions, full points will be awarded only for solutions that are correct, complete and sufficiently display concepts and methods of MATA32.
4. You may use one standard hand-held calculator. The following devices are forbidden: laptop computers, Blackberry or similar devices, cell-phones, I-Pods, MP-3 players or similar devices.
5. Extra paper, notes and textbooks are forbidden.

* (NOTE: ∵ there were two term tests for MATA32 Fall 2008, some questions/concepts on this Test #2 would not appear for a course with one midterm)

Do not write in the boxes below.

Info.	Part A
3	32

Part B

1	2	3	4	5	6
8	9	10	10	20	8

Total
100

Part A - Multiple Choice For each of the following, **carefully circle the letter** next to the response you think is correct. Each right answer earns 4 points and no answer/wrong answers earn 0 points. Justification is not required, but a small workspace is provided for your rough work.

1. The slope of the tangent line to the curve $y = \dfrac{x+5}{x^2}$ at the point where $x = 2$ is

   (a) $3/2$ (b) $-1/4$ (c) 2 (d) $-3/2$

$$y = \frac{1}{x} + \frac{5}{x^2} \rightarrow y' = -\frac{1}{x^2} - \frac{10}{x^3}$$

$$\therefore y'(2) = -\frac{1}{4} - \frac{10}{8} = -\frac{12}{8} = \boxed{-\frac{3}{2}}$$

2. If $p = -2q + 80$ is a demand equation where $0 < q < 40$, then we have unit elasticity at

   (a) no value of q (b) $q = 24$ (c) $q = 16$ (d) $q = 20$
   (e) a value of q not given by (a), (b), (c) or (d)

$$\eta = \frac{\frac{p}{q}}{p'(q)} = \frac{\frac{-2q+80}{q}}{-2} = \frac{-2 + \frac{80}{q}}{-2}$$

$$= 1 - \frac{40}{q}$$

$$|\eta| = 1 \Leftrightarrow \frac{40}{q} - 1 = 1 \Leftrightarrow \boxed{q = 20}$$

3. If y is defined implicitly as a function of x by the equation $x^2y + \ln(y) - 6x = -9$ then the value of $y\,'$ at $(3, 1)$ equals

(a) 2 (b) 0 (c) -2 (d) none of (a), (b) or (c).

$$2xy + x^2y' + \frac{y'}{y} - 6 = 0$$

@ (3,1) we get $6 + 9y' + y' - 6 = 0$

$$10y' = 0 \qquad \boxed{y' = 0}$$

4. For $x > 0$, let $f(x) = x^k$ where k is a constant and $0 < k < 1$. We may conclude that

(a) $f\,'(x)$ is decreasing on $(0, \infty)$ (b) $f\,'(x)$ is increasing on $(0, \infty)$

(c) $f(x)$ is decreasing on $(0, \infty)$ (d) none of (a), (b) or (c) is true

$$f'(x) = kx^{k-1}$$

$$f''(x) = \underbrace{k(k-1)}_{<0}\underbrace{x^{k-2}}_{>0} \Longrightarrow f''(x) < 0$$

so $\boxed{f'(x) \text{ is } \downarrow}$

5. If $u = (ex)^{\sqrt{x}}$ then $\left.\frac{du}{dx}\right|_{x=1}$ equals

(a) e (b) $\frac{3e}{2}$ (c) 1 (d) none of (a), (b) or (c)

$$\ln(u) = \sqrt{x}\,\ln(ex) = \sqrt{x}\,(1 + \ln(x))$$

$$\frac{u'}{u} = \frac{1}{2\sqrt{x}}(1 + \ln(x)) + \sqrt{x}\left(\frac{1}{x}\right)$$

$u(1) = e^1 = e$ so

$$u' = e\left[\frac{1}{2}(1+0) + 1\right] = \boxed{\frac{3e}{2} = \left.\frac{du}{dx}\right|_{x=1}}$$

6. If $h(t) = 2\left(3^t\right)$ then $h''(0)$ equals

(a) $(\ln(3))^4$ (b) 0 (c) $2(\ln(3))^2$ (d) $2\ln(9)$

$h'(t) = 2(3^t)\ln(3)$

$h''(t) = 2(3^t)(\ln(3))^2$

$\boxed{\therefore\ h''(0) = 2(\ln(3))^2}$

7. If the average cost to produce q number of units is given by $\bar{c} = \dfrac{800}{q+6}$ then the marginal cost of production at 14 units is

(a) -2 (b) 380 (c) 40 (d) 38 (e) not given by (a), (b), (c) or (d)

$$c = \bar{c}\cdot q = \frac{800q}{q+6}$$

$$c' = \frac{800(q+6) - 800q}{(q+6)^2}$$

$$c'(14) = \frac{800(20) - 800(14)}{(20)^2} = \frac{4800}{400} = 12$$

8. Exactly how many of the following four mathematical statements are true:

T (i) A rational function is discontinuous only at points where its denominator is equal to 0.

F (ii) A function g is differentiable at s if and only if g is both continuous and positive at s.

F (iii) The domain of a function H is the same at the domain of the function H'.

F (iv) If $f'(a) = 0$ or $f'(a)$ is undefined then f has a relative maximum or minimum at a.

(a) none (b) one (c) two (d) three (e) all four

Only (i) is true.

Part B - Full Solution Problem Solving

1. Find and simplify the expression $y'' + 2y' + y$ where $y = 5xe^{-x}$

[8 points]

$$y'' + 2y' + y = (5e^{-x} - 5xe^{-x})'$$
$$+ 2(5e^{-x} - 5xe^{-x}) + 5xe^{-x}$$
$$= (-5e^{-x} - 5e^{-x} + 5xe^{-x})$$
$$+ 10e^{-x} - 10xe^{-x} + 5xe^{-x}$$
$$= 0e^{-x} + 0xe^{-x}$$
$$= \boxed{0}$$

2. A manufacturer finds that when 2,500 calculators are produced per day, the average production cost is \$31.50 and the marginal production cost is \$8.35 Based on this data, approximate to two (correctly rounded) decimals

(a) the production cost of 2,501 calculators per day

[5 points]

$$\bar{C}(2500) = 31.50 = \frac{C(2500)}{2500}$$

$$C(2501) \approx C(2500) + C'(2500)$$
$$= (31.50)(2500) + 8.35$$
$$= \boxed{78{,}758.35}$$

(b) the marginal average production cost of 2,500 calculators per day

[4 points]

$$\bar{C}(q) = \frac{C(q)}{q} \Rightarrow (\bar{C}(q))' = \frac{C'(q)q - C(q)}{q^2}$$

$$\text{At } q = 2{,}500,\ (\bar{C}(2500))' = \frac{(8.35)(2500) - (2500)(31.50)}{(2500)^2}$$

$$\doteq -.00926$$

To 2 dp, $\boxed{-.01}$

3. In all of this question let $f(x) = \dfrac{x^{-1} + c^{-1}}{x^{-1} - c^{-1}}$ where $c > 0$ is a constant.

(a) Find the point(s) at which f is discontinuous. [3 points]

$$f(x) = \frac{\frac{1}{x} + \frac{1}{c}}{\frac{1}{x} - \frac{1}{c}}$$ so f is discontinuous at 0 and c

($\frac{1}{x}$ is not defined at 0 and $\frac{1}{x} - \frac{1}{c}$ is 0 when $x = c$)

(b) Find the value(s) of c for which $f'(3) = 1/2$ [7 points]

Write f as $$f(x) = \frac{\frac{c+x}{cx}}{\frac{c-x}{cx}} = \frac{c+x}{c-x} \; ; \; x \neq 0, c$$

$$\therefore f'(x) = \frac{(c-x) - (c-x)}{(c-x)^2} = \frac{2c}{(c-x)^2}$$

$$\frac{1}{2} = f'(3) = \frac{2c}{(c-3)^2} = \frac{2c}{c^2 - 6c + 9}$$

$$\Rightarrow \quad c^2 - 6c + 9 = 4c$$

$$c^2 - 10c + 9 = 0$$

$$(c-9)(c-1) = 0$$

$$\boxed{\therefore c = 9 \text{ or } 1}$$

$r^3 = 1 + 2r^2$

4. A positive real number r has the property that "its cube is one more than twice its square"

(a) State the cubic polynomial function $f(x)$ whose leading coefficient is 1 such that r is a root of f. [2 points]

$f(x) = x^3 - 2x^2 - 1$

(We have $f(r) = 0$)

(b) Justify mathematically why $r \in (2, 3)$ [3 points]

f is a polynomial, hence cts on $[2,3]$

$f(2) = -1 < 0$ $\quad f(3) = 8 > 0$

By IVT (Intermediate Value thm) or OST (opposite sign thm), $f(r) = 0$ for some $r \in (2,3)$.

(c) State the Newton method iteration formula and use it with $x_1 = 2$ to find x_2 and x_3. Use four decimals in your answers. [5 points]

Newton method iteration formula $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

With f above, $\quad x_{n+1} = x_n - \left[\dfrac{x_n^3 - 2x_n^2 - 1}{3x_n^2 - 4x_n}\right]$

$\boxed{x_1 = 2}$

$x_2 = 2 - \left[\dfrac{8 - 8 - 1}{12 - 8}\right] = 2 + \frac{1}{4} = \boxed{2.2500 = x_2}$

$x_3 = 2.25 - \left[\dfrac{(2.25)^3 - 2(2.25)^2 - 1}{3(2.25)^2 - 4(2.25)}\right]$

$\boxed{x_3 \approx 2.2071}$

5. In all of this question let $g(x) = x^{5/3} + 5x^{2/3}$ $= x^{2/3}(x+5)$

(a) Verify that $g'(x) = \dfrac{5(x+2)}{3x^{1/3}}$ [4 points]

$$g'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$

$$= \frac{5}{3}\,\frac{1}{x^{1/3}}\,(x+2) = \frac{5(x+2)}{3x^{1/3}} \checkmark$$

(b) State the critical values of g and determine the open intervals on which g is increasing or decreasing. Sufficiently justify your answer. [6 points]

$g'(-2) = 0$ so -2 is a critical #

g' is undefined at 0 so 0 is also a critical #. There are no others!

		$\boxed{-2}$		$\boxed{0}$	
$x+2$	$-$	0	$+$	$+$	$+$
$x^{1/3}$	$-$	$-$	$-$	0	$+$
$g'(x)$	$+$	0	$-$	u	$+$
$g(x)$	$\nearrow$	$-$	$\searrow$	$\vert$	$\nearrow$

g is increasing on $(-\infty, -2)$ & $(0, \infty)$

g is decreasing on $(-2, 0)$

(These conclusions are seen from the chart above.)

Question 5 continued

(c) Find the x-values of the relative extrema of g and the corresponding function values there. Round your answer(s) to two decimals. Sufficiently justify your solution.

[5 points]

By 1^{ST} DT, g has a relative max @ -2 of value $g(-2) = (-2)^{2/3}(3)$
≈ 4.76

Also by 1^{ST} DT, g has a relative min @ 0 of value $g(0) = 0$

(d) Find the absolute extrema of g on the closed interval $[-1, 1/2]$ $= I$

[5 points]

$\because$ g is continuous on the closed + bounded interval I, the Closed Interval Method for absolute extrema solves the problem.

Of the two critical #'s, only $0 \in I$

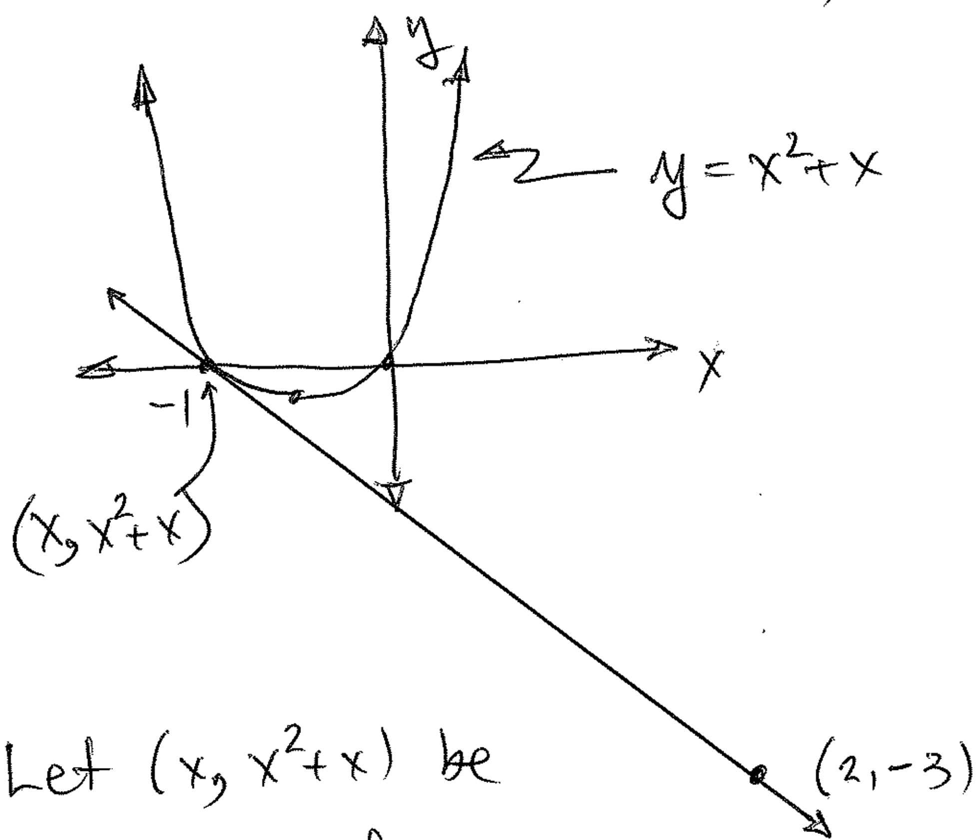
$g(0) = 0$ $\quad g(-1) = 4$ $\quad g(\frac{1}{2}) = (\frac{1}{2})^{2/3}(\frac{11}{2})$
≈ 3.46

Absolute max of g is 4 at $x = -1$

Absolute min of g is 0 at $x = 0$

(Absolute extrema of g are for $x \in I$)

6. Find the equation of the line having negative slope that passes through the point $(2, -3)$ and is tangent to the curve $y = x^2 + x$. $= x(x+1)$ [8 points]



(Clearly is not on $y = x^2 + x$)

Let (x, x^2+x) be a point of tangency

$$\text{Slope} = \frac{x^2+x+3}{x-2}$$

But: Slope = Derivative at (x, x^2+x)

$$\therefore \frac{x^2+x+3}{x-2} = 2x+1 \quad (= y')$$

$$x^2+x+3 = 2x^2-3x-2$$

$x^2-4x-5=0$ so $x = 5$ or $x = -1$

$y'(-1) = -1 < 0$ (but $y'(5) = 11 > 0$) $\therefore$ discard 5 ✓

$(y - 0 = -1(x-(-1)))$

$\boxed{\text{Equation of tangent line is } y = -x - 1}$