* * SOLUTIONS * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32 - Midterm Test - Calculus for Management I

Examiner: R. Grinnell

Date: February 19, 2011
Duration: 110 minutes
Time: 9:00 am

Clearly indicate the following information:

Last Name (Print): Solutions + Basic Stats

Given Name(s)(Print): ______________________

Student Number: ______________________

Signature: ______________________

Tutorial Number (e.g. TUT0032): ______________________

Carefully circle the name of your Teaching Assistant:

Bin XU

Sujanthan SRISKANDARAJAH

Read these instructions:

1. This test has 11 numbered pages. It is your responsibility to ensure that at the beginning of the test, all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are strongly encouraged to **write your test in pen or other ink**. Tests written in pencil will be denied any remarking or revision privilege.

* * * SOLUTIONS * * *

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
B	D	E	A	D	B	A

Do not write anything in the boxes below.

Info.	Part A
2	28

Part B

1	2	3	4	5
13	12	18	13	14

Total
100

The following formulas may be helpful:

$S = P(1+r)^n \qquad S = Pe^{rt} \qquad \text{revenue} = (\text{price})(\text{quantity})$

For ordinary annuity: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

For annuity due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2**. Each right answer earns 4 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. The value of $\lim_{x\to -1}\left(\frac{x^3+5x^2+4x}{2x^2-2} - e^{1+x}\right)$ is

   (A) $-7/4$ (B) $-1/4$ (C) $3/4$ (D) a number not in (A) - (C)
   (E) undefined

1st $\lim_{x\to -1} e^{1+x} = e^0 = 1$

2nd for $\lim_{x\to -1}$ we hope to factor $x+1 = x-(-1)$

$$\frac{x^3+5x^2+4x}{2x^2-2} = \frac{x(x+1)(x+4)}{2(x+1)(x-1)} \to \frac{-3}{-4} = \frac{3}{4} \text{ as } x\to -1$$

$\therefore$ answer $= \frac{3}{4} - 1 = -\frac{1}{4}$

2. What (3-decimal approximate) annual percentage rate of interest compounding semi-annually is equivalent to a 2.3% APR compounding continuously ?

   (A) 4.653 (B) 2.304 (C) 2.287 (D) 2.313 (E) none of (A) - (D)

$$\left(1+\frac{r}{2}\right)^2 = e^{.023}$$

$$1+\frac{r}{2} = \left(e^{.023}\right)^{1/2} = e^{.0115}$$

$$\therefore r = 2\left(e^{.0115}-1\right) \approx 2.313276$$

3. The present value of \$5,000 due five years from now if interest is 3.3% APR compounding monthly is (to the nearest dollar rounded up)

   (A) \$982 (B) \$4,932 (C) \$4,228 (D) \$4,359 (E) none of (A) - (D)

$$PV = 5{,}000\,(1.00275)^{-60} \approx 4{,}240.4287$$

$$\frac{3.3}{(100)(12)} = .00275$$

4. The slope of the curve $y = \dfrac{4+x}{1+2x} - \dfrac{1}{\sqrt{x}}$ at the point where $x = 1$ is

(A) $-5/18$ (B) $5/18$ (C) $-23/18$ (D) $-11/6$
(E) a number not in (A) - (D) (F) nonexistent

$$y' = \frac{1(1+2x) - (4+x)(2)}{(1+2x)^2} + \frac{1}{2x^{3/2}}$$

$$y'(1) = \frac{3-10}{9} + \frac{1}{2} = -\frac{7}{9} + \frac{1}{2} = -\frac{5}{18}$$

5. If $w = 3y^5 - 2y^3 + 1$ and $y = 2x^2 - 1$ and $x = 3t - 2$ then $\dfrac{dw}{dt}$ when $t = 1$ equals

(A) 2 (B) 24 (C) 120 (D) 108 (E) none of (A) - (D)

$$\frac{dw}{dt} = \frac{dw}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dt} = (15y^4 - 6y^2)(4x)(3)$$

$t=1 \rightarrow x=1 \rightarrow y=1$ $\quad \left.\dfrac{dw}{dt}\right|_{t=1} = (9)(4)(3) = 108$

6. If the demand equation for a manufacturer's product is $p = \dfrac{50q+30}{\sqrt{q}}$ dollars per unit when q units are sold, then the marginal revenue when $q = 25$ is

(A) 753 (B) 378 (C) 500 (D) 4.76 (E) none of (A) - (D)

$$r = pq = 50q^{3/2} + 30q^{1/2}$$

$$r'(q) = 75q^{1/2} + 15q^{-1/2}$$

$$r'(25) = 75(5) + \frac{15}{5} = 378$$

7. The least number of months it takes an investment to increase by exactly 12% at 3.6% APR compounding three times annually is

(A) 40 (B) 39 (C) 38 (D) 32 (E) a number not in (A) - (D)
(F) uncertain, because we do not know the value of the initial investment

Solve for t where

$$1.12P = P(1.012)^{3t}$$

$$t = \ln(1.12)/3[\ln(1.012)]$$

$t \approx 3.1668696$ (years)

≈ 3 years $+ 2.00244$ months

$\therefore$ We must take our answer as 40 months.

Make sure that your answers are printed in the letter boxes at the top of page 2

Reasoning:

(3 times annual compounding $\Rightarrow$ Compounding only after each 4 month period)

Part B (Full Solution Questions) Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32.

1. In all of this question let $f(x) = \sqrt{8x^2 + 2x}$ $= (8x^2+2x)^{1/2}$

   (a) Use the usual rules of differentiation to find $f'(1)$ [5 points]

$$f'(x) = \frac{1}{2}(8x^2+2x)^{-1/2} \cdot (16x+2)$$

$$f'(1) = \frac{18}{2\sqrt{10}} = \boxed{\frac{9}{\sqrt{10}}}$$

   (b) Use the definition of derivative (i.e. "first principles") to find $f'(1)$ [8 points]

$$f'(1) = \lim_{h\to 0}\left[\frac{f(1+h)-f(1)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sqrt{8(1+h)^2+2(1+h)} - \sqrt{10}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sqrt{8(1+h)^2+2(1+h)} - \sqrt{10}}{h} \cdot \frac{\sqrt{8(1+h)^2+2(1+h)} + \sqrt{10}}{\sqrt{8(1+h)^2+2(1+h)} + \sqrt{10}}\right]$$

$$= \lim_{h\to 0} \frac{8(1+h)^2 + 2(1+h) - 10}{h\left(\sqrt{8(1+h)^2+2(1+h)} + \sqrt{10}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{8} + 16h + 8h^2 + \cancel{2} + 2h - \cancel{10}}{h\left(\sqrt{8(1+h)^2+2(1+h)} + \sqrt{10}\right)}$$

$$= \lim_{h\to 0} \frac{8h^2 + 18h}{h\left(\sqrt{8(1+h)^2+2(1+h)} + \sqrt{10}\right)}$$

$$= \lim_{h\to 0} \frac{18}{\sqrt{8(1+h)^2+2(1+h)} + \sqrt{10}} = \frac{18}{2\sqrt{10}} = \boxed{\frac{9}{\sqrt{10}}}$$

(exactly as above!)

2. A total debt of \$6,000 due 3 years from now and \$5,000 due 4.75 years from now is to be repaid by three payments as follows:

   The first payment of \$3,000 is made 3 months from now

   A second payment is made 2 years from now

   The third payment (which is 80% of the second) is paid 40 months from now.

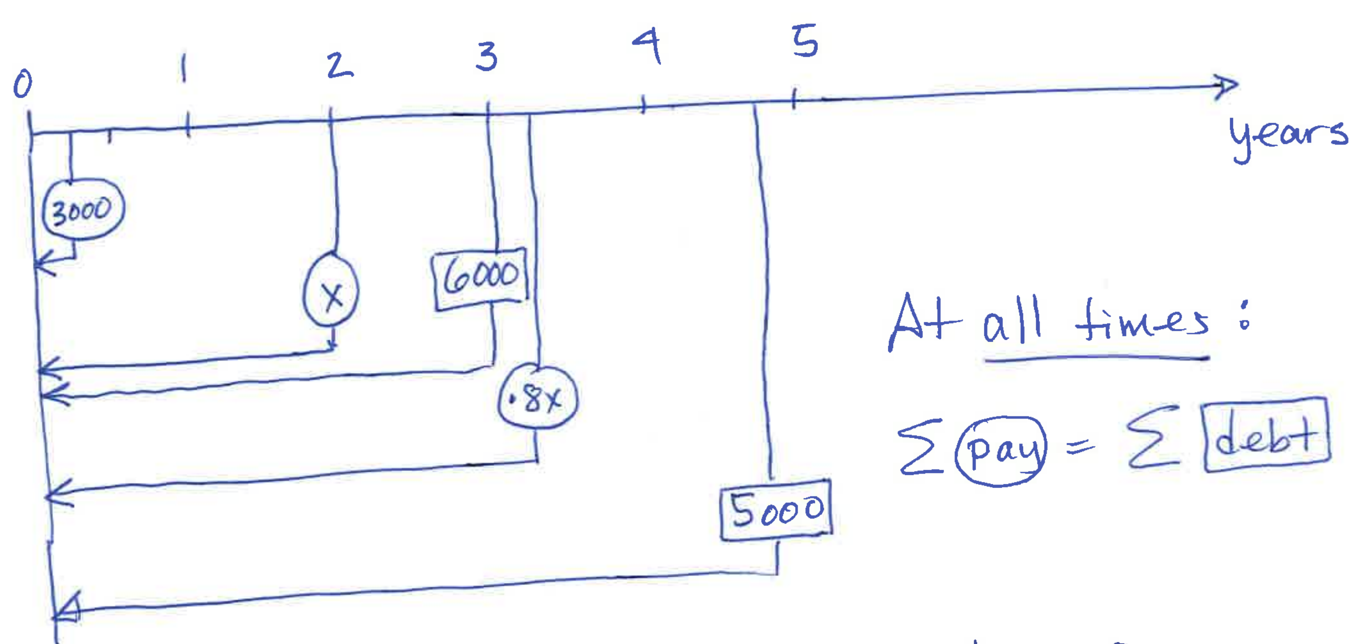
   Interest is 3.12% APR compounding monthly. Find the amount of the second and third payments. Round your final answers up to the nearest dollar. A money-time diagram and a well-labeled equation of value is required for full points. [12 points]

Let x be the amount of the 2^{nd} payment

$\therefore$ 3^{rd} payment $= .8x$

$$r = \frac{3.12}{(100)(12)} = .0026 = \text{periodic rate}.$$

In the money-time diagram: $\boxed{\text{debt}}$, pay. All in \$



At all times:

$$\sum \text{pay} = \sum \boxed{\text{debt}}$$

Equation of value: calibrate to time 0.

$$3000(1.0026)^{-3} + x(1.0026)^{-24} + (.8x)(1.0026)^{-40}$$
$$= 6000(1.0026)^{-36} + 5000(1.0026)^{-57}$$

$$1.66066045x = 6799.946756$$

$$x \approx 4094.723853$$

ANSWER:

2^{nd} pay $= 4095$

3^{rd} pay $= 3276$

3. In all of this question let $u(x) = \dfrac{6}{2x-1}$ if $x \geq 1$ and $u(x) = \dfrac{17}{3} + \dfrac{x^2}{3}$ if $x < 1$.

(a) Use the definition of continuity to show that u is continuous at $x = 1$. [5 points]

We must check that

$$\lim_{x\to 1} u(x) = u(1) \quad (*)$$

$$u(1) = \frac{6}{2-1} = 6$$

$$\lim_{x\to 1^-} u(x) = \lim_{x\to 1^-}\left(\frac{17}{3} + \frac{x^3}{3}\right) \quad \text{(poly} \therefore \text{cts)}$$

$$= \frac{17}{3} + \frac{1}{3} = \underline{\underline{6}}$$

$$\lim_{x\to 1^+} u(x) = \lim_{x\to 1^+}\left(\frac{6}{2x-1}\right)$$

$$= \underline{\underline{6}} \quad \left(\begin{array}{c}\text{rational} \\ \text{cts @ 1}\end{array}\right)$$

Work below shows

$$\lim_{x\to 1^-} u(x) = 6 = \lim_{x\to 1^+} u(x)$$

$\therefore$ $(*)$ is true.

(b) Find the equation of the tangent line to the curve $y = u(x)$ at the point where $x = -2$ [7 points]

General form is $y - u(-2) = u'(-2)(x+2)$

$$u(-2) = \frac{17}{3} + \frac{(-2)^2}{3} = \frac{21}{3} = 7$$

$$u'(x) = \left(\frac{17}{3} + \frac{x^2}{3}\right)' = \frac{2}{3}x \Rightarrow u'(-2) = -\frac{4}{3}$$
$(x < 1)$

$$-\frac{8}{3} + \frac{21}{3} = \frac{13}{3}$$

Equation of tangent $\boxed{y - 7 = -\frac{4}{3}(x+2)}$

or $\boxed{y = -\frac{4}{3}x + \frac{13}{3}}$

or $\boxed{4x + 3y = 13}$

Any are ✓

(Question 5 continued)

(c) Find the point(s) on the curve $y = u(x)$ at which the tangent line is parallel to the line you found in part (b). [6 points]

parallel $\Leftrightarrow$ equal slope $\Leftrightarrow$ equal derivative

We consider solving the eq$^{\underline{n}}$ $u'(x) = -\frac{4}{3}$ (Part (b))

One solution is exactly from (b) : $x = -2$

For other possibilities, we consider solving

$u'(x) = -\frac{4}{3}$ where $x \geq 1$.

When $x \geq 1$, $u(x) = \frac{6}{2x-1} \Rightarrow u'(x) = \frac{-12}{(2x-1)^2}$

Solving $\frac{-12}{(2x-1)^2} = -\frac{4}{3}$

$\therefore (2x-1)^2 = 9 \rightarrow 2x-1 = \pm 3$

$+3$ case : $2x-1 = 3 \rightarrow x = 2$ ok ✓

-3 case : $2x-1 = -3 \rightarrow x = -2$ not ok ... we are assuming $x \geq 1$

$\therefore$ the only sol$^{\underline{n}}$ to $u'(x) = -\frac{4}{3}$ and $x \geq 1$ is $x = 2$. Then $u(2) = \frac{6}{3} = 2$

$\therefore$ the desired point is $\boxed{(2,2)}$

4. In all of this question consider the following 3-year car lease that has two parts:

(i) a "one time" damage fee of $675 and

(ii) a monthly fee of $297

(a) If the monthly fee is subject to 2.4% APR and there is no interest on the damage fee, calculate the lump sum payment of the entire lease if it is made at the beginning of the 3-year period and all monthly fees are paid at the start of each month. Round your final answer up to the nearest dollar. [6 points]

LS = lump sum payment

$$= 675 + 297 + 297\left[\frac{1-(1.002)^{-35}}{.002}\right]$$

$\approx 11{,}001.8380$

$\left(r = \frac{2.4}{(100)(12)} = .002\right)$

$\boxed{\text{Take LS as } \$11{,}002}$

(b) If the monthly fee and damage fee are subject to 1.8% APR, calculate the lump sum payment of the entire lease if it is made at the end of the 3-year period and all monthly fees are paid at the end of each month. Round your final answer up to the nearest dollar. [7 points]

LS = lump sum payment

$r = \frac{1.8}{(100)(12)} = .0015$

$$= 675(1.0015)^{36} + 297\left[\frac{(1.0015)^{36}-1}{.0015}\right]$$

$\approx 11{,}689.9192$

$\boxed{\text{Take LS as } \$11{,}900}$

5. The following information is used through all of this question. A manufacturer finds that, for a production of 9 units, the cost is \$670 and the marginal cost is ~~\$30.~~ \$21

(a) Find the approximate cost of producing 10 units [4 points]

Let $c(q)$ = cost function, q = quantity

$c(10) \approx c(9) + c'(9)$

$= 670 + 21 = \boxed{691}$

(b) Assume the cost function has the form $c(q) = Aq - \dfrac{B}{q+1} + K$ where A, B, and K are constants and $q \geq 0$ is the number of items produced. Assume also that:

(i) $\lim_{q\to\infty} a(q) = 20$ where $a(q)$ is the average cost function, and

(ii) the marginal cost at $q = 4$ is 24.

Find the value of A, B, and K and state the final cost function $c(q)$. [10 points]

$a(q) = \dfrac{c(q)}{q} = A - \dfrac{B}{q(q+1)} + \dfrac{K}{q} \longrightarrow A$ as $q \to \infty$

But $\lim_{q\to\infty} a(q) = 20$ $\quad \boxed{\therefore A = 20}$

Thus $c(q) = 20q - \dfrac{B}{q+1} + K$

$c'(q) = 20 + \dfrac{B}{(q+1)^2}$ From (a), $c'(9) = 21$ (or $c'(4) = 24$ either is ok)

$\therefore 21 = 20 + \dfrac{B}{100} \longrightarrow \boxed{B = 100}$

From (a), $c(9) = 670$

Thus $670 = 20(9) - \dfrac{100}{10} + K \rightarrow \boxed{K = 500}$

$\therefore K = 670 - 180 + 10$

$\boxed{\text{Final cost function} \quad c(q) = 20q - \dfrac{100}{q+1} + 500}$

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Basic Test Stats

N = 144 students wrote the test

$\overline{x}$ = approx average = 59%

Pass % ≈ 72.2 % ($\frac{104}{144}$ students passed)

Approximate % # of students per decile %

100	0
90's	2.8
80's	9.0
70's	17.4
60's	22.2
50's	21.0
40's	10.4
30's	12.5
20's	3.5
10's	0.7
1's	0.7

very unusual