*** SOLUTIONS + STATISTICS ***

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32F - Calculus for Management I - Midterm Test

Examiners: R. Grinnell, R. Haslhofer, E. Moore

Date: October 10, 2015
Duration: 110 minutes
Time: 9:00 am

Last Name (PRINT BIG) ______________________

First Name(s) (PRINT BIG) ______________________

Student Number ______________________

Signature ______________________

Carefully circle your TA name and tutorial number

Fazle CHOWDHURY 8	Ke TONG 5 12
Ruixue DAI 13	Tianqi WANG 25
Taylor ESCH 7 14	Dexiang (Dexter) WU 4 20
Rui (Ray) GAO 23 24	Binya XU 1 2
Yaodong (Terry) GAO 3 6 9	Huiyan XU 17 21
Martin HO 10 22	Ruoqi YU 18 19
Nicholas LEBLANC 15 16	

Statistics on Page 7

Read these instructions

1. This test has 11 numbered pages. Check that all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of Page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or Page 11. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator of any make or model. All other electronic devices, scrap paper, notes, textbooks, pen/pencil carrying cases, and foods are forbidden at your workspace (either visibly or in any sort of carrying case or by accident). You may have a drink that is not in any kind of paper cup or a container with a paper label.
4. You are encouraged to write your test in pen or other ink. If any questions (Part A or B) or rough work is displayed in pencil, then your entire test will be denied any remarking privilege.

Print letters for Part A (Multiple Choice Questions) in these boxes

1	2	3	4	5	6	7
C	E	A	B	E	F	D

Do not write anything in the boxes below

Info.	Part A
4	21

Part B

1	2	3	4	5	6
12	11	14	11	11	16

Total
100

Some formulas

$$S = P(1+r)^n \qquad S = Pe^{rt}$$

$$S = R\left[\frac{(1+r)^n - 1}{r}\right] \qquad A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

$$S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R \qquad A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is correct in the boxes at the top of Page 2**. Each right answer earns 3 points and no answer/wrong answers earn 0 points. No justification is required.

1. The present value (rounded-up to the nearest dollar) of \$400 due in 5 years at 6% APR compounding semiannually is

   (A) \$224 (B) \$225 (C) \$298 (D) \$299 (E) \$329 (F) \$346

$$r = \frac{.06}{2} = .03$$

$$PV = 400(1.03)^{-10}$$

$$\approx 297.76 \quad \uparrow 298$$

2. If $f(x) = \dfrac{8x^4 + 2x}{4\sqrt{x}} + \dfrac{x^{3/2}}{2}$ then the value of $f'(1)$ is

   (A) 7.5 (B) 8.75 (C) 6 (D) 3.75 (E) 8 (F) none of (A) - (E)

$$f(x) = 2x^{7/2} + \frac{1}{2}x^{1/2} + \frac{1}{2}x^{3/2}$$

$$f'(x) = 7x^{5/2} + \frac{1}{4}\frac{1}{x^{1/2}} + \frac{3}{4}x^{1/2}$$

$$f'(1) = 7 + \frac{1}{4} + \frac{3}{4} = 8$$

3. What lump-sum amount now (rounded-up to the nearest dollar) would generate \$200 payments at the beginning of each month for two years if interest is 4.8% APR compounding monthly?

   (A) \$4,587 (B) \$4,567 (C) \$4,547 (D) \$4,527 (E) none of (A) - (D)

0 1 2 . . . 23 24

200 200 200 200

$$r = \frac{.048}{12} = .004$$

$$PV = 200\left[\frac{1-(1.004)^{-23}}{.004}\right] + 200$$

$$\approx 4,586.37 \quad \uparrow 4,587$$

$\ln(1)=0$

4. The value of $\lim_{x\to -2}\left(\frac{8x^2+16x}{x^3-4x}+\ln(x+3)\right)$ is (A) 2 (B) -2 (C) 4

(D) -3 (E) a number not in (A) - (D) (F) none of (A) - (E)

$$\frac{8x^2+16x}{x^3-4x} = \frac{8x(x+2)}{x(x+2)(x-2)} = \frac{8x}{x(x-2)}, \; x \neq -2$$

$$\rightarrow \frac{-16}{(-2)(-4)} = \frac{-16}{8} = -2$$

5. If the average revenue from selling q units of a product is given by $\bar{r} = \frac{800}{q+3}+6$, then the marginal revenue when 7 units are sold is

(A) 24 (B) -8 (C) 14 (D) 66 (E) 30 (F) none of (A) - (E)

$$\bar{r} = \frac{800}{q+3}+6 \rightarrow r = (\bar{r})(q) = \frac{800q}{q+3}+6q \text{ (revenue)}$$

$$r' = \frac{(800)(q+3)-800q}{(q+3)^2}+6 \qquad r'(7) = \frac{8000-5600}{100}+6$$

$$= \frac{2400}{100} = 24+6 = 30$$

6. If $y = x^2\sqrt{3x^2+4}$ then $\frac{dy}{dx}$ when $x=2$ is

(A) $\frac{1}{2}$ (B) 6 (C) $\frac{33}{2}$ (D) 14 (E) 28 (F) 22

$$y' = 2x\sqrt{3x^2+4} + (x^2)\left(\tfrac{1}{2}\right)(3x^2+4)^{-1/2}(6x)$$

$$y'(2) = (4)(4) + \left(\tfrac{4}{2}\right)\left(\tfrac{1}{4}\right)(12) = 16+6 = 22$$

7. The least number of months it takes a principal to increase by 13% at a nominal rate of 2.4% compounding quarterly is

(A) 21 (B) 24 (C) 66 (D) 63 (E) 62 (F) 84

$n = $ # of compounds $\qquad r = \frac{.024}{4} = .006$

$$1.13\cancel{P} = \cancel{P}(1.006)^n$$

$$n = \frac{\ln(1.13)}{\ln(1.006)} \approx 20.43$$

Round # of compounds up to get 21.

$\therefore$ MONTHS $= (21)(3) = 63$

Make sure your answers are printed in the letter boxes at the top of Page 2

 (3 months/compound)

Part B (Full Solution Questions) Show all of your work. Answers/solutions will earn full points only if they are correct, complete, and sufficiently display relevant concepts from MATA32F.

1. The two parts of this question are independent of each other.

(a) Find $\frac{dy}{dx}$ in fully factored form where $y = (4x+3)^3(2x+5)^6$. [6 points]

$$\frac{dy}{dx} = 3(4x+3)^2(4)(2x+5)^6 + (4x+3)^3 6(2x+5)^5(2)$$

$$= (4x+3)^2(2x+5)^5\left[12(2x+5) + 12(4x+3)\right]$$

$$= (4x+3)^2(2x+5)^5\left[24x+60+48x+36\right]$$

$$= (4x+3)^2(2x+5)^5\left[72x+96\right]$$

$$\frac{dy}{dx} = \boxed{24(4x+3)^2(2x+5)^5(3x+4)}$$

(b) Let $f(x) = \dfrac{x^3}{\ln(x)}$. Find the slope-intercept form of the equation of the tangent line to the graph of $y = f(x)$ at the point where $x = e$. Express your answer in terms of mathematical constants, not decimals. [6 points]

$$f(e) = \frac{e^3}{\ln(e)} = e^3 \quad \therefore \text{ Point is } (x_0, y_0) = (e, e^3)$$

$$f'(x) = \frac{3x^2\ln(x) - x^3\left(\frac{1}{x}\right)}{[\ln(x)]^2} \quad \text{(No need to simplify!)}$$

$$\therefore f'(e) = \frac{3e^2 - e^2}{[\ln(e)]^2} = 2e^2 \quad \text{Slope} = m = 2e^2$$

$$y - y_0 = m(x - x_0)$$

$$y - e^3 = 2e^2(x - e)$$

$$y = 2e^2x - 2e^3 + e^3 = \boxed{2e^2x - e^3 = y}$$

Payment ☐
Debt ○

○ 12months ○

2. A total debt of \$6,000 due one year from now and \$4,000 due 40 months from now is to be repaid by three payments as follows:

☐ (i) a first payment at the end of six months from now;

☐ (ii) a second payment at the end of 28 months from now that is 90% of the first payment;

☐ (iii) a third payment that is \$900 more than the second payment and is made five years from now. 60 months.

Interest is 4.8% APR compounding monthly. Find the amount of the three payments. Carry at least five decimals in all of your calculations. Round your final answers up to the nearest dollar. A complete money-time diagram and equation of value are required for full points.

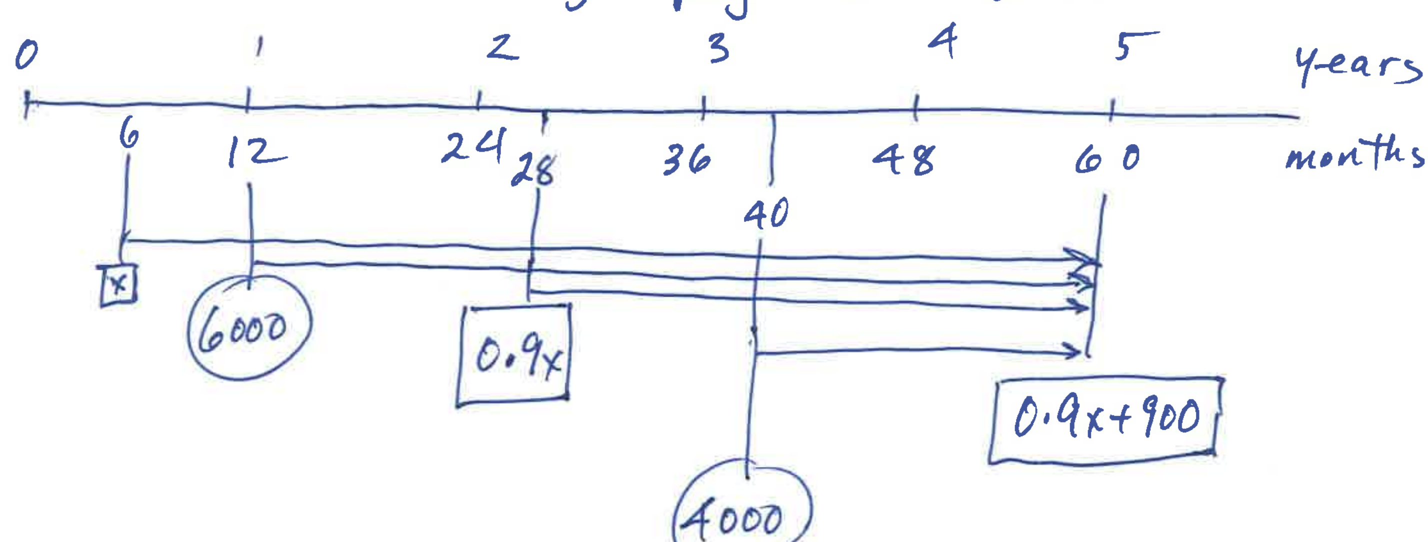
[11 points]

$r = \frac{.048}{12} = .004$

Let x represent 1^{ST} payment.

$\therefore$ 2^{nd} payment $= 0.9x$

3^{rd} payment $= 0.9x + 900$



Calibrate to time = End of 60 months.

Equation of value: $\boxed{\text{Value of all Pay}}$ = Value of all Debt [@ 60 months]

$$x(1.004)^{54} + (0.9x)(1.004)^{32} + (0.9x+900)(1.004)^{0}$$

$$= (6000)(1.004)^{48} + 4000(1.004)^{20}$$

$$1.2405678x + 1.0226367x + 0.9x + 900$$

$$\approx 7{,}267.239368 + 4{,}332.456865$$

$$3.1632045x \approx 10{,}699.69623$$

$$x \approx 3{,}382.59964$$

$\uparrow$ $x \approx 3{,}383$

$$\boxed{\begin{array}{l} 1^{ST} \approx \$3{,}383 \\ 2^{nd} = \$3{,}045 \\ 3^{rd} = \$3{,}945 \end{array}}$$

3. The three parts of this question are independent of each other.

(a) Evaluate $\lim_{x \to -\infty} \frac{6x}{\sqrt{9x^2+7}}$ [3 points]

$$= \lim_{x \to -\infty} \frac{6x}{\sqrt{9x^2\left(1+\frac{7}{x^2}\right)}}$$

$$= \lim_{x \to -\infty} \frac{6x}{\sqrt{9x^2}\sqrt{1+7/x^2}}$$

$$= \lim_{x \to -\infty} \frac{6x}{(-3x)\sqrt{1+7/x^2}}$$

$\sqrt{9x^2} = -3x$ as x is <0

$$= \boxed{-2}$$

(b) Is there a value of the constant c that makes $f(x) = \begin{cases} 2x-1 & if \quad x>1 \\ c & if \quad x=1 \\ x^2+1 & if \quad x<1 \end{cases}$ continuous $x=1$? If so, find it. If not, justify why not. [4 points]

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1) = 1$$

$2 \neq 1$ $\therefore \lim_{x \to 1} f(x)$ DNE

$\boxed{\therefore \text{ No } c \text{ exists}}$

(c) Find the point(s) on the graph of $y = \frac{2x}{1+3x}$ where the slope of the tangent line is $\frac{1}{2}$. [7 points]

$$y' = \frac{2(1+3x) - 2x(3)}{(1+3x)^2}$$

$$= \frac{2}{(1+3x)^2}$$

Solve $\frac{2}{(1+3x)^2} = \frac{1}{2}$

$(1+3x)^2 = 4$

$1+3x = \pm 2 \Rightarrow x = \frac{1}{3}$

$x = -1$

$$y\left(\tfrac{1}{3}\right) = \frac{\frac{2}{3}}{1+3\left(\frac{1}{3}\right)} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

$$y(-1) = \frac{-2}{1-3} = \frac{-2}{-2} = 1$$

$\boxed{\text{Points are } \left(\tfrac{1}{3}, \tfrac{1}{3}\right) \text{ and } (-1, 1)}$

4. The two parts of this question are independent of each other.

(a) Find the maximum amount of compound interest (as a percentage of a given principal) that can be earned over 8 years with 7% APR. Round your final percentage answer down to the nearest integer. [5 points]

MAX amount $\leftrightarrow$ continuously compounding interest.

P = principal   A = amount

$$A = Pe^{(.07)(8)} = Pe^{.56}$$

$$\left(\begin{array}{c}\text{Compound Interest} \\ \text{as \% of } P\end{array}\right) = \left(\frac{A-P}{P}\right) \times 100 = \left[\frac{Pe^{.56} - P}{P}\right] \times 100$$

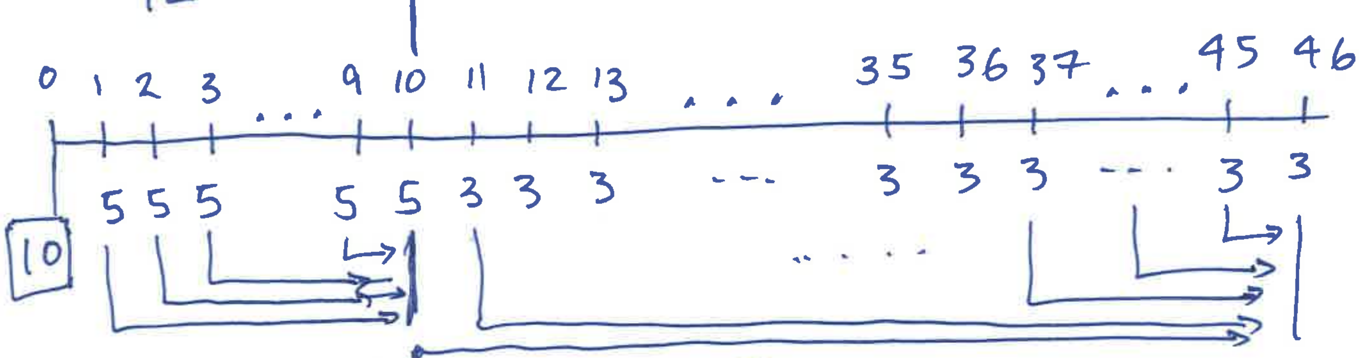
$\boxed{\text{ANSWER } 75\%}$ $\approx 75.067 \ (\downarrow)$

(b) Assume an interest rate of 6% APR compounding monthly for your savings account which has $1,000 in it. Your parents electronically put $500 into the account at the end of every month for 10 months. Immediately after the tenth payment, they instantly reduce this to $300 which they put into your account at the end of every month for the next three years. How much will be in the account the moment after the final deposit? Round your final answer down to the nearest dollar. [6 points]

ALL MONEY IN $100

$$r = \frac{.06}{12} = .005$$

3 years = 36 months



$$FV = 10(1.005)^{46} + 5\left[\frac{(1.005)^{10} - 1}{.005}\right](1.005)^{36}$$

$$+ 3\left[\frac{(1.005)^{36} - 1}{.005}\right]$$

$\approx 12.5788 + 61.1984 + 118.0083$

$= 191.7855$

(\$100's) $(\downarrow)$

$\boxed{\text{ANSWER } \$19{,}178}$

5. The two parts of this question are independent of each other.

(a) Let $g(x) = (x^2 - 1)e^{-x}$. Find all value(s) of x for which $g'(x) = 0$. Express your answer(s) in terms of mathematical constants, not decimals. [6 points]

$$g'(x) = 2xe^{-x} + (x^2-1)e^{-x}(-1)$$
$$= e^{-x}\left[2x - (x^2-1)\right]$$
$$= e^{-x}\left[-x^2+2x+1\right]$$

$g'(x) = 0 \iff -x^2+2x+1 = 0$ because $e^{-x} > 0$.

Solve $x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$$\boxed{\text{ANSWER } x = 1 \pm \sqrt{2}}$$

(b) Assume $P(x)$ and $Q(x)$ are differentiable functions and that for all real numbers x, $P(Q(x)) = x$ and $P'(x) = 4 + \left[\frac{P(x)}{2}\right]^2$. Find $Q'(0)$. [5 points]

Differentiate

$$P'(Q(x)) \cdot Q'(x) = 1$$

$P(Q(0)) = 0$

(Let $x=0$)

$$P'(Q(0)) \cdot Q'(0) = 1$$

But: $P'(Q(0)) = 4 + \left[\frac{P(Q(0))}{2}\right]^2 = 4 + \left[\frac{0}{2}\right]^2 = 4$

$$\therefore Q'(0) = \frac{1}{P'(Q(0))} = \frac{1}{4}$$

$$\boxed{\text{ANSWER } Q'(0) = \frac{1}{4}}$$

6. In all of this question let $\bar{c} = \dfrac{324}{\sqrt{q^2+35}} + \dfrac{5}{q} + \dfrac{19}{18}$ be an average cost function (in \$100 per unit) where $q > 0$ is quantity (i.e. production) per day.

(a) Find the marginal cost when 17 units are produced per day. [7 points]

Cost function is $c = (\bar{c})(q) = \dfrac{324q}{\sqrt{q^2+35}} + 5 + \dfrac{19}{18}q$

$$c'(q) = \frac{324\sqrt{q^2+35} - (324q)\frac{1}{\cancel{2}}(q^2+35)^{-1/2}(\cancel{2}q)}{(q^2+35)} + \frac{19}{18}$$

$$c'(17) = \frac{(324)(18) - \frac{324(17)^2}{18}}{324} + \frac{19}{18} = 3$$

$\boxed{\text{ANSWER } \$300}$

(b) Assume hypothetically that q is very, very large. Show that for such values of q, the average cost is always between \$105 and \$107 per unit. [4 points]

For $q > 0$ we have $\bar{c}(q) > \frac{19}{18} \approx 1.0\dot{5} > 1.05$

$\lim_{q \to \infty} \bar{c}(q) = \frac{19}{18}$. Thus, if q is very, very large, eventually, we have that $\bar{c}(q) < 1.07$ $\therefore$ $\$105 < \bar{c}(q) < \$107$.

(c) Assume $p(q) = r(q) - c(q)$ where p is a profit function and r is a revenue function. Moreover, when $q = 17$, assume the revenue is \$50,000 and the marginal revenue is \$800 per unit. Estimate the profit when $q = 18$. Round your answer down to the nearest dollar. [5 points]

$p(18) - p(17) \approx p'(17)$

$\therefore p(18) \approx p'(17) + p(17)$

$= r'(17) - c'(17) + r(17) - c(17)$

$c(17) = (17)\bar{c}(17) = (17)(19.349\dot{6}) = 328.94$

$= 8 - 3 + 500 - 328.9\dot{4}$

$= 5 + 171.0\dot{5} = 176.0\dot{5}$

$\boxed{\text{ANSWER } \$17,605}$

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Simple Test Statistics

of students who wrote test on Sat Oct 10 = 706

Date all statistics below were compiled = Sun Nov 15

# of students still enrolled in MATA32F on Sun Nov 15 = 636

($\therefore$ 70 students dropped course)

# of students out of 636 who wrote the test = $\boxed{618 = N}$

($\therefore$ 18 students were still enrolled and did not write test)

All % and averages are taken out of $N = 618$

Decile	#	% of 618 (≈)
100	0	0
90's	22	3.6%
80's	74	12%
70's	100	16.2%
60's	107	17.3%
50's	99	16%
40's	77	12.5%
30's	56	9.1%
20's	42	6.8%
10's	32	5.2%
1's	9	1.5%
	618	≈ 100%

Highest Score = 98.5%

Lowest Score = 1.5%

Test average over $N = 618$ students is ≈ 56.6%

- Test average ignoring scores below 20% is ≈ 59.7%
- % ≥ 50% on test is ≈ 65%
- % ≥ 60% on test is ≈ 49%
- % ≥ 70% on test is ≈ 31.7%
- % ≥ 80% on test is ≈ 15.5%