* SOLUTIONS * * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32 - Midterm Test - Calculus for Management I

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Date: October 22, 2010
Time: 7:00 pm
Duration: 110 minutes

Clearly indicate the following information:

Last Name (Print): ____________________

Given Name(s)(Print): ____________________

Student Number: SOLUTIONS

Signature: (and Test Statistics ... p. 11)

Tutorial Number (e.g. TUT0032): ____________________

Carefully circle the name of your Teaching Assistant:

Shibing CHEN	Duy Minh DANG	Yiwen (Louis) LUO
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Srishta CHOPRA	Zhou LI	Zheng WANG
Fazle CHOWDHURY	Yik Chau (Kry) LUI	Bin XU

Read these instructions:

1. This test has 11 numbered pages. It is your responsibility to ensure that at the beginning of the test, all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are strongly encouraged to **write your test in pen or other ink**. Tests written in pencil will be denied any remarking or revision privilege.

* * SOLUTIONS * *

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
c	b	a	d	a	c	b

Do not write anything in the boxes below.

3 points iff ALL information is given on page 1

Info.	Part A
3	
3	21

Part B

1	2	3	4	5	6
13	17	7	12	12	15

Total
100

The following formulas may be helpful:

$$S = P(1+r)^n \qquad S = Pe^{rt}$$

For ordinary annuity: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

For annuity due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Test Statistics are on page 11

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. The slope of the tangent line to the curve $y = 6\sqrt[3]{x}$ at the point $(x, 12)$ on the curve is

   (a) 1.5 (b) $2^{1/3}$ (c) 0.5 (d) about 0.382
   (e) a number that is not in (a) - (d) (f) uncertain, since we do not know the value of x

$y = 6x^{1/3}$ (When $y = 12$, $12 = 6\sqrt[3]{x}$ $\therefore x = 8$)

$y' = 2x^{-2/3}$ We find $y'(8)$

$y'(8) = 2(8)^{-2/3}$
$= 2(2)^{-2}$
$= \boxed{0.5}$

2. The value of $\lim_{x\to 3}\left(\frac{3x^2+12x-63}{-x^3+3x^2-x+3} + e^{3-x}\right)$ is

   (a) 0 (b) -2 (c) $e^2 - 3$ (d) 0.4 (e) none of (a) - (d)

$\lim_{x\to 3} e^{3-x} = e^{3-3} = 1$

$3x^2+12x-63 = 3(x+7)\cancel{(x-3)} \longrightarrow \frac{3(10)}{-10} = -3$
$-x^3+3x^2-x+3 = -(x^2+1)\cancel{(x-3)} \longrightarrow$

$(-3)+1 = \boxed{-2}$

3. What (3-decimal approximate) annual percentage rate of interest compounding semi-annually is equivalent to a 3% APR compounding monthly ?

   (a) 3.019 (b) 3.194 (c) 3.419 (d) 4.528 (e) none of (a) - (d)

Rate $= r$ Solve $\left(1+\frac{r}{2}\right)^2 = \left(1+\frac{.03}{12}\right)^{12}$

$\frac{.03}{12} = .0025$ $1+\frac{r}{2} = (1.0025)^6$

$r = 2[(1.0025)^6 - 1]$

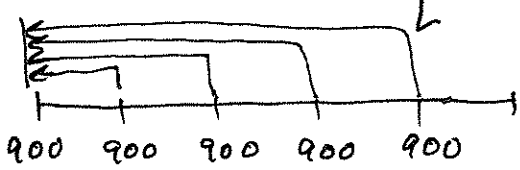
≈ 3.0188

$\approx \boxed{3.019}$

4. The present value of an annuity consisting of five \$900 deposits, each paid at the beginning of the month, at a rate of 3% APR compounding monthly is (approximately)

(a) \$4,182 (b) \$4,286 (c) \$4,338 (d) \$4,478 (e) none of (a) - (d)

$$PV = 900 + 900\left[\frac{1-(1.0025)^{-4}}{.0025}\right] \approx 4{,}477.61$$



$\boxed{4{,}478}$

5. The value of the constant a that makes $\lim_{x\to-\infty}\left(\frac{-ax^2+8x}{x^2+6x+9}+\sqrt{\frac{x}{x+1}}\right) = -4$ is

(a) 5 (b) 3 (c) -5 (d) -3 (e) a number not in (a) - (d) (f) nonexistent

We want $-a+1=-4$

so $\boxed{a=5}$

6. If $y = \left(\frac{4x^2-x}{\sqrt{x}}\right)^3$ then the value of $\left.\frac{dy}{dx}\right|_{x=1}$ is

(a) 27 (b) 135/2 (c) 297/2 (d) 167 (e) none of (a) - (d)

$$y = \left(4x^{3/2} - x^{1/2}\right)^3$$

$$y' = 3\left(4x^{3/2}-x^{1/2}\right)^2\cdot\left(6x^{1/2}-\tfrac{1}{2}x^{-1/2}\right)$$

$$y'(1) = 3(3)^2\left(6-\tfrac{1}{2}\right) = (27)\left(\tfrac{11}{2}\right) = \boxed{\frac{297}{2}}$$

7. The least number of months it takes an investment to increase by exactly 32% at 2.4% APR compounding quarterly is ← (11 years + 9 months)

(a) 140 (b) 141 (c) 144 (d) 557 (e) a number not in (a) - (d)
(f) uncertain, because we do not know the value of the initial investment

t = years Solve $1.32 = (1.006)^{4t}$ $\qquad \frac{.024}{4} = .006$

$$t = \frac{\ln(1.32)}{4\ln(1.006)} \approx 11.602 \text{ years}$$

11.602 years $\approx$ 139.224 months $\approx$ 11 years 7.224 months

Make sure that your answers are printed in the letter boxes at the top of page 2

HOWEVER: Compounding is quarterly, so we must round up to the next quarter year (i.e. 9 months) $\therefore$ ANSWER = $\boxed{\text{11 years + 9 months}}$

Part B (Full Solution Questions) Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32.

1. (a) Suppose you have a sum of money to invest and you are considering a choice of two rates: (A) 3.6% APR compounding quarterly or (B) 3.58% APR compounding daily. Which of (A) or (B) is the better of the two rates for your investment? Sufficiently justify your answer. Note: 365 days = 1 year Look @ effective rates [4 points]

$$r_e(A) = \left(1 + \frac{.036}{4}\right)^4 - 1 \approx 3.6488922\ \%$$

$$r_e(B) = \left(1 + \frac{.0358}{365}\right)^{365} - 1 \approx 3.644671595\%$$

$r_e(A) > r_e(B)$ so (A) is better.

(b) Find the APR for which, over a 10-year period, the maximum amount of compound interest is 32.41% Express your answer as a percentage, rounded to 2 decimals (i.e. in the form $X.YZ\%$) [5 points]

Let $P > 0$ be an arbitrary principal

Maximum amount of compound interest occurs precisely when compounding is continuous.

Solve for r: $\left(\frac{Pe^{10r} - P}{P}\right) \times 100 = 32.41$

$$r = \frac{\ln(1.3241)}{10} \approx 0.028073 \qquad \boxed{\text{APR is } 2.81\%}$$

(c) Upon graduating from UTSC, you get a job as a junior financial advisor that has a starting annual salary of \$57,000. You are informed that your annual salary will increase by a constant percentage of the current year's salary, that each annual salary increase will occur at the end of the year, that you will have five salary increases, and your salary after the fifth increase will be \$85,000. Calculate the constant percentage salary increase (rounded to 2 decimals so that your answer has the form $X.YZ\%$) [4 points]

Let r represent the constant annual salary increase.

Solve for r: $85 = 57(1+r)^5$

$$r = \left(\frac{85}{57}\right)^{1/5} - 1 \approx .0832004$$

$$\boxed{r \approx 8.32\%}$$

2. In all of this question let $f(x) = \dfrac{48}{1+3x}$ $= 48(1+3x)^{-1}$

(a) Use the rules of differentiation to find $f'(x)$ [3 points]

$$f'(x) = -48(1+3x)^{-2}(3) = \frac{-144}{(1+3x)^2}$$

(b) Find all point(s) (x, y) on the curve $y = f(x)$ for which the tangent line is parallel to the line $9x + y = 5$ [8 points]

$y = -9x + 5 \rightarrow$ slope $= -9$

Parallel $\rightarrow$ solve $f'(x) = -9$

$$\frac{-144}{(1+3x)^2} = -9$$

$(1+3x)^2 = 16$

$\therefore 1+3x = 4$ or $1+3x = -4$

$\therefore x = 1$ $\quad$ $\therefore x = -\frac{5}{3}$

$f(1) = 12$

$f(-\frac{5}{3}) = \frac{48}{1-5} = -12$

$\boxed{\text{Points are } (1, 12) \text{ and } (-\tfrac{5}{3}, -12)}$

(c) Use the definition of derivative (i.e "first principles") to find $f'(x)$ [6 points]

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{\frac{48}{1+3x+3h} - \frac{48}{1+3x}}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{48 + 144x - 48 - 144x - 144h}{h(1+3x+3h)(1+3x)}\right]$$

$$= \lim_{h \to 0} \frac{-144}{(1+3x+3h)(1+3x)} = -\frac{144}{(1+3x)^2}$$

3. In all of this question k is a constant and

$$u(x) = \begin{cases} \ln(2x-1) - k & if \quad x \geq 1 \\ k^2 x - 2 & if \quad x < 1 \end{cases}$$

Find the value(s) of k that make u continuous at $x = 1$. Justify your solution completely.

[7 points]

u is continuous at $x = 1$ if and only if

$$\lim_{x \to 1} u(x) = u(1)$$

$$u(1) = \ln(1) - k = -k$$

We also need $\lim_{x \to 1^-} u(x) = u(1)$

So, $\lim_{x \to 1^-} (k^2 x - 2) = -k$

$\therefore \quad k^2 - 2 = -k$

Solve $\quad k^2 + k - 2 = 0$

$(k+2)(k-1) = 0$

$\therefore k = -2$ or 1

$\boxed{k = -2 \text{ or } 1}$

4. A total debt of $\$2,000$ due 3 years from now and $\$7,500$ due 55 months from now is to be repaid by 3 payments as follows:

   The first payment is made 6 months from now.

   The second payment is $\$3,000$ and is made 2 years from now.

   The third payment is 50% more than the first payment and is made 4 years from now.

   Interest is 4.8% APR compounding monthly. Find the amount of the first and third payments. Round your final answers for these amounts up to the nearest dollar. A money-time diagram and an equation of value is required for full points. [12 points]

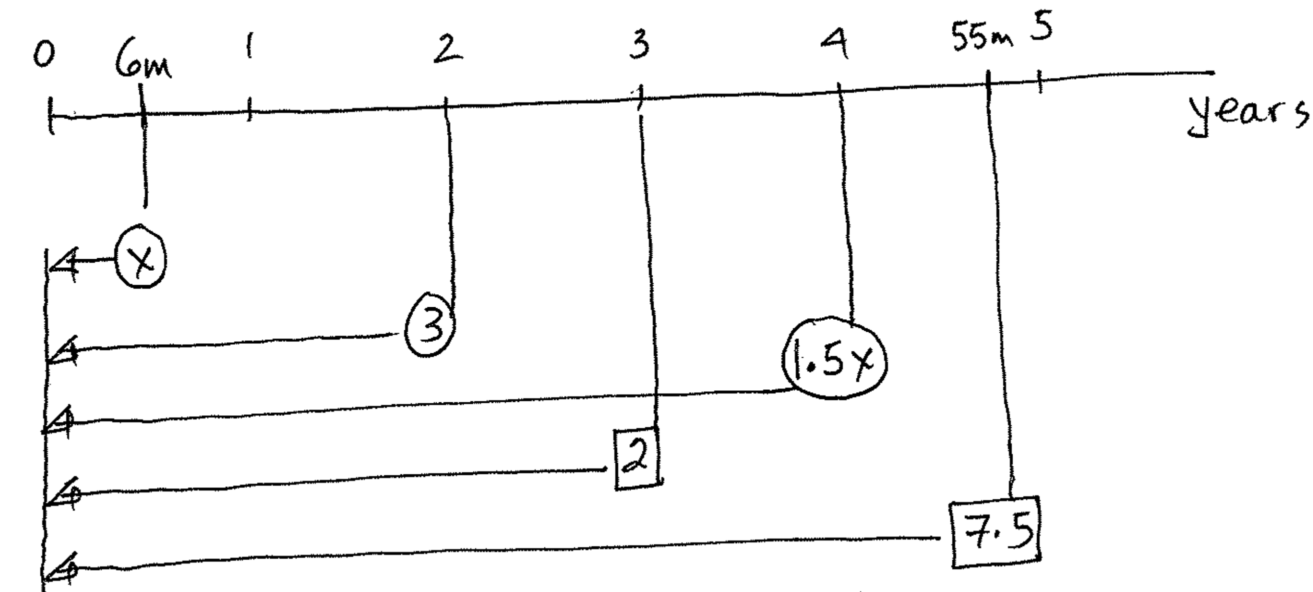
$r = \frac{.048}{12} = .004$ debt in $\boxed{}$ pay in ◯

m = months

All monies in \$1,000's

Let x represent amount of 1st payment.

Money-time diagram calibrated to time = 0 (i.e. "now")



Equation of value: (all in months)

$$x(1.004)^{-6} + 3(1.004)^{-24} + (1.5x)(1.004)^{-48} = 2(1.004)^{-36} + 7.5(1.004)^{-55}$$

$$x(.976332448 + 1.238434506) \approx 1.732272993 + 6.021532835 - 2.725913982$$

$$x \approx 2.270167449$$

1st payment \$2,271

3rd payment \$3,406

5. In all of this question a is a positive constant and $f(x) = \dfrac{x^{-1}+a^{-1}}{x^{-2}-a^{-1}}$ $= \dfrac{\frac{1}{x}+\frac{1}{a}}{\frac{1}{x^2}-\frac{1}{a}}$

(a) Find all points of discontinuity of f [3 points]

f is discontinuous only where it is undefined

$\boxed{x = 0, \pm\sqrt{a}}$

(this is because f is ultimately a rational $f^{\underline{n}}$)

(b) Write f as a rational function. [3 points]

$$f(x) = \frac{x^2a\left(\frac{a+x}{xa}\right)}{x^2a\left(\frac{a-x^2}{x^2a}\right)}$$

$$= \frac{ax+x^2}{a-x^2}$$

$$\boxed{f(x) = \frac{x^2+ax}{-x^2+a}}$$

(c) Find $f'(1)$ and simplify. [6 points]

$$f'(x) = \frac{(a+2x)(a-x^2)-(ax+x^2)(-2x)}{(a-x^2)^2}$$

$$\therefore f'(1) = \frac{(a+2)(a-1)-(a+1)(-2)}{(a-1)^2}$$

$$= \frac{a^2+a-2+2a+2}{(a-1)^2}$$

$$\boxed{f'(1) = \frac{a^2+3a}{(a-1)^2} = \frac{a(a+3)}{(a+1)^2}}$$

Either "ok"

6. The following information is used through all of this question. A manufacturer finds that when 8 units are produced, the average cost per unit is \$64 and the marginal cost is \$18

(a) Find the approximate cost of producing 9 units. [4 points]

Let c = cost to produce q units

Given is $\bar{c}(8) = 64 \longrightarrow c(8) = \bar{c}(8) \cdot 8$

$= 64 \times 8 = 512$

Lectures: $c(q+1) \approx c(q) + c'(q)$

$\therefore c(9) \approx c(8) + c'(8) = 512 + 18 = 530$

$\boxed{\text{Cost is about } \$530}$

(b) Calculate the marginal average cost to produce 8 units. [5 points]

$\bar{c}(q) = \frac{c(q)}{q} \rightarrow [\bar{c}(q)]' = \frac{c'(q) \cdot q - c(q)}{q^2}$

$\therefore [\bar{c}(q)]'\Big|_{q=8} = \frac{c'(8) \cdot 8 - c(8)}{64} = \frac{(18)(8) - 512}{64}$

$= \boxed{-5.75}$

(c) Find the cost function assuming that it is a quadratic and the fixed cost is \$400 [6 points]

Let $c(q) = Aq^2 + Bq + K$ where

A, B, K are constants to be found.

Fixed cost $= 400 = c(0) = K \quad \therefore \underline{K = 400}$

$c(8) = 512 = 64A + 8B + 400 \quad \therefore 8A + B - 14 = 0$ (1)

$c'(q) = 2Aq + B \quad c'(8) = 18 = 16A + B$

$\therefore B = 18 - 16A$ (2)

Sub (2) into (1): $8A + 18 - 16A - 14 = 0$

$-4 - 8A = 0 \quad \underline{A = \frac{1}{2}} \quad \underline{B = 10}$

$\boxed{\therefore c(q) = \frac{q^2}{2} + 10q + 400}$