* * SOLUTIONS + STATISTICS * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32F (Calculus for Management I) Midterm Test

Examiners: R. Buchweitz
R. Grinnell

Date: October 20, 2014
Duration: 110 minutes
Time: 5:00 pm

Last Name (PRINT BIG) * SOLUTIONS + STATISTICS *

First Name(s) (PRINT BIG) ______________________

Student Number ______________________

Signature ______________________

Carefully circle your TA name and tutorial number

Fazle CHOWDHURY 8 21

Xiaopeng (Michelle) CUI 19 25

Taylor ESCH 7

Rui GAO 17 18

Yaodong GAO 1 14

Anran JIA 22 23

Namhee (Terry) KANG 2

Peiying LI 10

Pourya MEMARPANAHI 24

Ushya SHANMUGARAJAH 9 12

Zhendong SHAO 15 16

Chao (Jerry) SHEN 5 13

Xin (Aaron) SITU 3 6

Binya XU 20

Qianqian ZHU 4

Read these instructions

1. This test has 11 numbered pages. You should check that all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or page 11. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator of any make or model. All other electronic devices, extra paper, notes, textbooks, pen/pencil carrying cases, and foods are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are encouraged to write your test in pen or other ink. If any questions are answered in pencil, then your entire test is denied any remarking privilege.

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Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
A	D	B	E	B	C	C

Do not write anything in the boxes below.

Info.	Part A
3	21

Part B

1	2	3	4	5	6
16	12	12	11	12	13

Total
100

Some formulas

$$S = P(1+r)^n \qquad S = Pe^{rt} \qquad \eta = \frac{p/q}{\frac{dp}{dq}}$$

Ordinary: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

Due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. No justification is required.

1. If $u = (e^2x)^{\sqrt{x}}$ then $u'(1)$ equals $u(1) = e^2$

   (A) $2e^2$ (B) $2e$ (C) e^2 (D) $2e^2+1$ (E) none of (A) - (D)

$$\ln(u) = \sqrt{x}\ln(e^2x) = \sqrt{x}[\ln(e^2) + \ln(x)]$$

$$\frac{u'}{u} = \frac{1}{2\sqrt{x}}[2 + \ln(x)] + \sqrt{x}\left[0 + \frac{1}{x}\right]$$

$$\therefore u'(1) = \frac{e^2}{2}[2] + e^2 = \boxed{2e^2}$$

2. Which APR compounding quarterly is most closely equivalent to 2.68% APR compounding semi-annually?

   (A) 2.563% (B) 2.762% (C) 2.602% (D) 2.671% (E) 2.598%

Let a = desired APR.

Solve $\left(1+\frac{a}{4}\right)^4 = \left(1+\frac{.0268}{2}\right)^2$

$$1+\frac{a}{4} = \sqrt{1+\frac{.0268}{2}}$$

$$\therefore \frac{a}{4} = \sqrt{1+\frac{.0268}{2}} - 1$$

$$\therefore a = 4\left[\sqrt{1+\frac{.0268}{2}} - 1\right]$$

$$\approx \boxed{2.671}$$

3. The least whole number of months it takes a principal to increase by one-fifth at a nominal rate of 3.6% compounding three times annually is

   (A) 48 (B) 64 (C) 84 (D) 63 (E) 68 (F) none of (A) - (E)

Let n = # of compounding periods (i.e. # of 4-month periods)

" P = arbitrary principal

Consider $1.2\cancel{P} = \cancel{P}\left(1+\frac{.036}{3}\right)^n$

Solve for n:

$$\ln(1.2) = n\ln(1.012)$$

$$\therefore n = \frac{\ln(1.2)}{\ln(1.012)} \approx 15.284$$

Must round n up!

$\therefore$ Get $n = 16$

$\therefore$ # of months $= 16 \times 4 = \boxed{64}$

$= y'$

4. If y is defined implicitly by the equation $e^{xy} + y = 2 + (x+1)^2$ then the value of $\frac{dy}{dx}$ evaluated
at $(0, 2)$ is (A) $e - 1$ (B) $1/2$ (C) 1 (D) 2 $\boxed{\text{(E) } 0}$

$x = 0$
$y = 2$

$$e^{xy}(y + xy') + y' = 2(x+1)$$

$$e^{0}(2+0) + y' = 2 \Rightarrow \boxed{y' = 0}$$

$$\therefore 2 + y' = 2$$

5. The value of $\lim\limits_{x \to -1} \dfrac{4x^3 + 4x^2}{x^3 + x^2 + x + 1}$ is (A) -2 $\boxed{\text{(B) } 2}$ (C) 1 (D) 3 (E) 0

$$= \lim_{x \to -1} \frac{4x^2(x+1)}{x^2(x+1) + 1(x+1)}$$

$$= \lim_{x \to -1} \frac{4x^2}{x^2+1} = \frac{4}{2} = \boxed{2}$$

6. If $f(x) = 4x^2\sqrt{4x+1} + 0.4x$ then $f'(6)$ is

(A) 224.6 (B) 296 $\boxed{\text{(C) } 298}$ (D) 355.6 (E) none of (A) - (D)

$$f'(x) = 8x\sqrt{4x+1} + 4x^2 \tfrac{1}{2}(4x+1)^{-1/2}(4) + .4$$

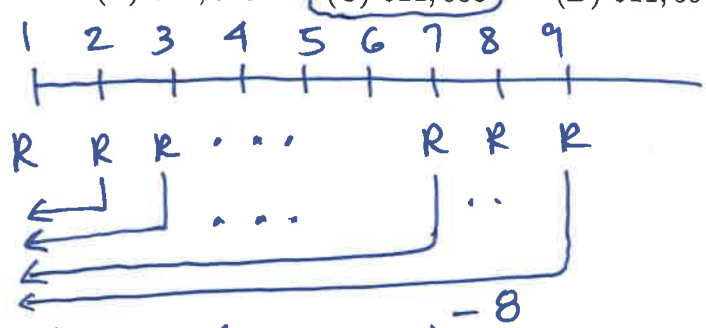
$$f'(6) = 48(5) + \frac{2(36)4}{5} + .4 = 240 + 57.6 + .4$$

$$= \boxed{298}$$

7. A student pays rent of \$1,300 a month due at the beginning of each month. Interest is 2.4% APR compounding monthly. What would be the total amount payable (rounded up to the nearest dollar) at the beginning of September, 2014 if the student wanted to pay nine months of rent in advance?

(A) \$12,884 (B) \$11,818 (C) \$11,608 (D) \$11,597 (E) none of (A) - (D)

Diagram is helpful!



$R = 1{,}300$

Round up to nearest \$
11, 608

$$A = 1300\left[\frac{1 - (1 + .002)^{-8}}{.002}\right] + 1300 \approx 11{,}607.02$$

***** Make sure your answers are printed in the letter boxes at the top of page 2*****

Part B (Full Solution Questions) Show all of your work. Answers/solutions will earn full points only if they are correct, complete, and sufficiently display relevant concepts from MATA32F.

1. In all of this question let $f(x) = \dfrac{3x+2}{7x+1}$.

   (a) Find $f'(x)$ and simplify. [4 points]

$$f'(x) = \frac{3(7x+1) - (3x+2)7}{(7x+1)^2} = \frac{21x + 3 - 21x - 14}{(7x+1)^2} = \boxed{\frac{-11}{(7x+1)^2}}$$

   (b) Find the equation of the tangent line to the curve $y = f(x)$ that is parallel to the line $11x + 64y = 75$ and whose point of tangency (x, y) satisfies $x, y > 0$. Give your answer in the form $Ax + By = C$ where A, B, C are integers and $A > 0$. [7 points]

Parallel → equal slope

$11x + 64y = 75$

→ $64y = -11x + 75$

$\therefore\ y = \boxed{-\frac{11}{64}}x + \frac{75}{64}$

Solve: $\dfrac{-11}{(7x+1)^2} = -\dfrac{11}{64}$

Get $x = 1$ or $x = -\frac{9}{7}$

Only keep $x = 1$

$f(1) = \frac{5}{8} = y$

$y - \frac{5}{8} = -\frac{11}{64}(x-1)$

$64y - 40 = -11x + 11$

$\boxed{11x + 64y = 51}$ is the desired eq^{n}

   (c) Assume g is a differentiable function such that $g(0) = g(2) = g'(2) = 2$ and $g'(0) = 4$. Find $\dfrac{dA}{dx}$ when $x = 0$ where $A(x) = g(x)g(f(x))$. [5 points]

$$\frac{dA}{dx} = A'(x) = g'(x)g(f(x)) + g(x)g'(f(x))f'(x)$$

When $x = 0$, $A'(0) = g'(0)g(f(0)) + g(0)g'(f(0))f'(0)$

$$= 4g(2) + 2g'(2)(-11)$$

$$= 4(2) + 2(2)(-11)$$

$$= 8 - 44 = \boxed{-36}$$

48 months

2. A total debt of \$9,000 due four years from now and \$6,000 due 68 months from now is to be repaid by three payments as follows:

   (i) a first payment at the end of the first year; 12 months

   (ii) a second payment at the end of 30 months from now and is 60% more than the first payment;

   (iii) a third payment that is 10% less than the second payment and is made five years from now. 60 months

   Interest is 2.4% APR compounding monthly. Find the amount of the three payments. Carry at least five decimals in all of your calculations. Round your final answers up to the nearest dollar. A complete money-time diagram and equation of value are required for full points.

   [12 points]

Let x = amount of 1ST payment.

pay

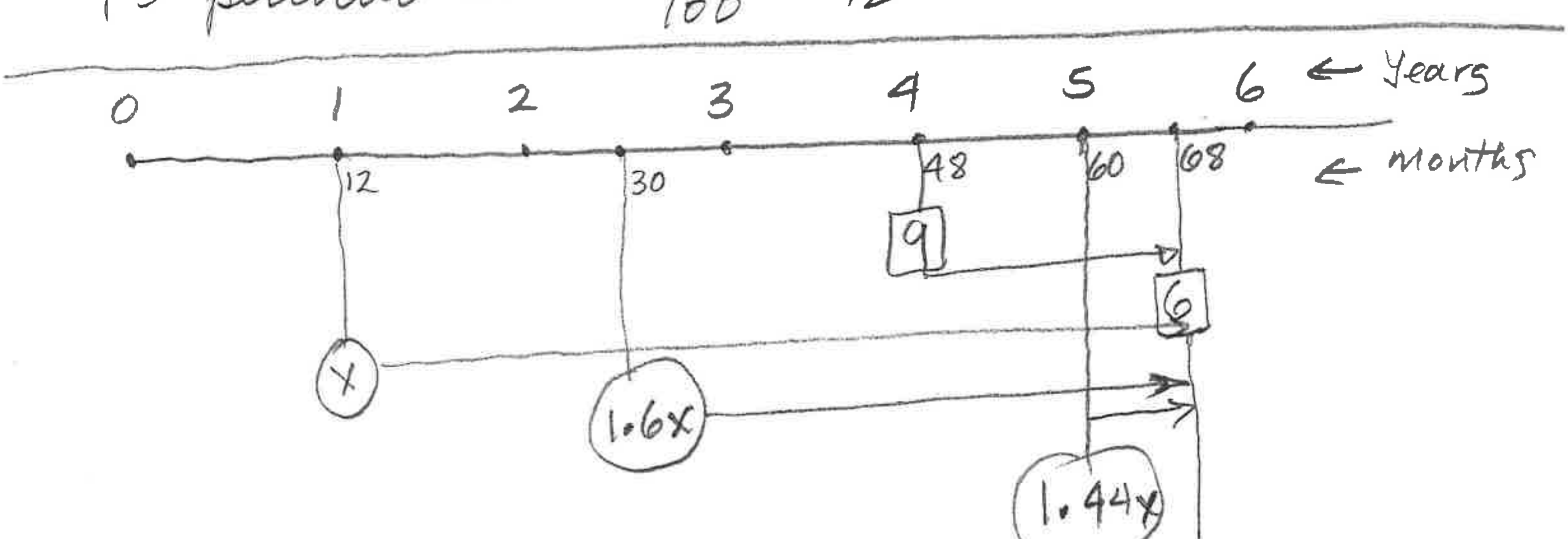
$\boxed{\text{debt}}$

$\therefore$ 2nd payment $= 1.6x$

3rd payment $= (.9)(1.6)x = 1.44x$

All money in \$1,000's

r = periodic rate $= \frac{2.4}{100} \times \frac{1}{12} = .002$



Calibrate to 68 months:

Pay

$\boxed{\text{Debt}}$

$$x(1.002)^{56} + 1.6x(1.002)^{38} + 1.44x(1.002)^{8} = 6 + 9(1.002)^{20}$$

$$x[1.11838776 + 1.726209096 + 1.463201927] = 15.36692278$$

$$x = \frac{15.36692278}{4.307798783} = 3.567233187$$

$\therefore$ 1ST $= 3,568$ 2nd $= 5,708$ 3rd $= 5,137$

3. For each limit below, evaluate it or determine that it does not exist. Use the ∞ or $-\infty$ symbol where appropriate.

(a) $\displaystyle\lim_{x\to-\infty}\left(\frac{\sqrt{9x^4+3x^2+1}}{x^2-6}+4e^{1/x}\right)$ [4 points]

$$= \lim_{x\to-\infty}\left(\frac{\sqrt{9x^4\left(1+\frac{3x^2}{9x^4}+\frac{1}{9x^4}\right)}}{x^2\left(1-\frac{6}{x^2}\right)}+4e^{1/x}\right) \qquad e^0=1$$

$$= \lim_{x\to-\infty}\left(\frac{3x^2}{x^2}\frac{\sqrt{1+\cancel{1/(3x^2)}+1/(9x^4)}}{\left(1-\cancel{6/x^2}\right)}+4e^{\frac{1}{x}}\right)$$

$$= 3+4 = \boxed{7}$$

(b) $\displaystyle\lim_{x\to1^-}\frac{x^2-1}{x-2\sqrt{x}+1}$ [4 points]

$$= \lim_{x\to1^-}\frac{x^2-1}{(\sqrt{x}-1)^2}$$

$$= \lim_{x\to1^-}\frac{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)(x+1)}{\cancel{(\sqrt{x}-1)}(\sqrt{x}-1)}$$

$$= \lim_{x\to1^-}\frac{(\sqrt{x}+1)(x+1)}{\sqrt{x}-1} = \boxed{-\infty \text{ and DNE}}$$

$x\to1^-$
$\Rightarrow x\approx1,\ x<1$
$\therefore \sqrt{x}\approx1,\ \sqrt{x}<1$
$\Rightarrow \sqrt{x}-1\approx0,\ <0$

(c) $\displaystyle\lim_{x\to e}\frac{x\ln(x)-e}{x-e}$ We evaluate usi g the definition of derivative

(A solution using l'Hopital's rule or with no justification will earn 0 points) [4 points]

Let $f(x)=x\ln(x)$ $\quad f(e)=e\ln(e)=e$

$f'(x)=\ln(x)+\frac{x}{x}$ $\quad f'(e)=\ln(e)+\frac{e}{e}=2$

$$\lim_{x\to e}\frac{x\ln(x)-e}{x-e}=\lim_{x\to e}\frac{f(x)-f(e)}{x-e}$$

$$=f'(e)=\boxed{2}$$

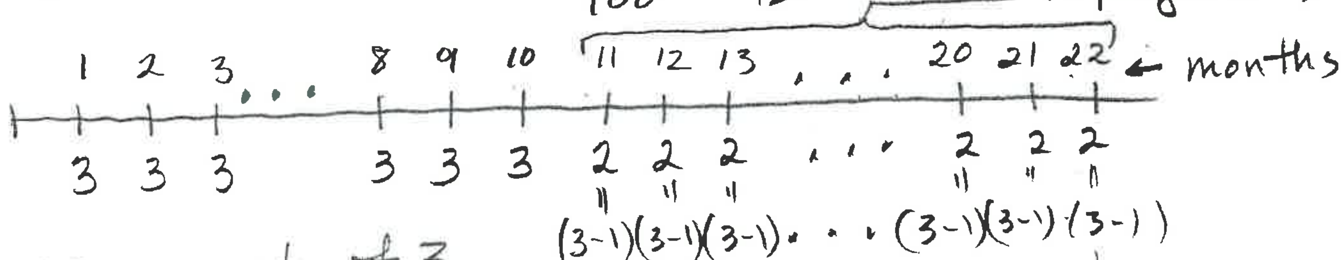
All money in \$100

4. The two parts of this question are independent of each other.

(a) In all of this question interest is 2.4% APR compounding monthly. Find the $\boxed{\text{future value}}$ of a generalized annuity consisting of \$300 payments at the end of each month for ten months and \$200 thereafter at the end of each month for one year. Round your final answer up to the nearest dollar. 12 months [7 points]

Diagram is very useful!

$$\text{Periodic rate} = r = \frac{2.4}{100} \times \frac{1}{12} = .002$$



• 10+12 = 22 payments of 3
• subtract 12 payments of 1

$$FV = 3\left[\frac{(1.002)^{22} - 1}{.002}\right] - 1\left[\frac{(1.002)^{12} - 1}{.002}\right]$$

$$= 3[22.46821895] - 12.13288398$$

$$= 55.271773$$

$$\boxed{FV = \$5,528}$$

(b) Since you are a UTSC Management student, your tablet is a target for on-line pop-up financial advertisements. Suppose one such advertisement makes the claim that, with a sufficiently high annual compounding rate, it is possible to earn 30% compound interest on an arbitrary investment of \$$P$ over exactly 3,003 days with an APR of 3.00%. Should you believe this claim? Sufficiently justify your answer with an appropriate calculation. [4 points]

Max possible compound interest is with continuous compounding. $t = \frac{3003}{365}$ years

Consider:

$$\left(\frac{Pe^{(.03)\left(\frac{3003}{365}\right)} - P}{P}\right) \times 100 \approx 27.995\% < 30\%$$

∴ We should not believe the advertisement as it is greater than max possible compound interest.

5. In all of this question let $f(x) = x^2 e^{-3x}$.

(a) Find all point(s) on the curve $y = f(x)$ where the tangent line is horizontal. [7 points]

$$f'(x) = 2xe^{-3x} + x^2 e^{-3x}(-3)$$

$$= e^{-3x}(2x - 3x^2)$$

$$f'(x) = 0 \text{ iff } 2x - 3x^2 = 0$$

$$\Leftrightarrow x(2-3x) = 0 \Leftrightarrow x = 0 \text{ or } x = \frac{2}{3}$$

$$f(0) = 0 \qquad f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 e^{-3\left(\frac{2}{3}\right)} = \frac{4}{9e^2}$$

$\therefore$ Points are $\boxed{(0,0) \text{ and } \left(\frac{2}{3}, \frac{4}{9e^2}\right)}$

(b) Find the function h such that $f''(x) = h(x)f(x)$. [5 points]

$$f''(x) = e^{-3x}(-3)(2x - 3x^2) + e^{-3x}(2 - 6x)$$

$$= (-6x + 9x^2 + 2 - 6x)e^{-3x}$$

$$= (9x^2 - 12x + 2)e^{-3x}$$

$$= \left(\frac{9x^2 - 12x + 2}{x^2}\right) x^2 e^{-3x}$$

$$= \left(\frac{9x^2 - 12x + 2}{x^2}\right) f(x)$$

$$\boxed{\therefore h(x) = \frac{9x^2 - 12x + 2}{x^2}}$$

$$\overline{C} = \frac{C}{q}$$

6. The parts of this question are independent of each other.

$AC = \overline{C}$

(a) Let $c = f(q)$ be a cost function where $q > 0$ is quantity. Assume when $q = 4$, the average cost is 80 and the marginal cost is 48.

(i) Estimate $c(5)$. $MC = c'$ [3 points]

$$C(5) \approx C(4) + C'(4)$$
$$= \overline{C}(4)\cdot 4 + 48 = (80)4 + 48 = \boxed{368}$$

(ii) Find the marginal average cost when $q = 4$. [4 points]

$(\overline{C})'$

$$(\overline{C})'\Big|_{q=4} = \left(\frac{C}{q}\right)'\Big|_{q=4} = \frac{C'(4)\cdot 4 - C(4)}{4^2}$$
$$= \frac{(48)(4) - 320}{16}$$
$$= \frac{192-320}{16} = \boxed{-8}$$

(b) Verify that the equation $x^3 = 5x^2 - 3$ has a solution, r, in the interval $[0, 1]$. Use Newton's method to find the approximation x_2 to r. Begin by selecting the appropriate starting value x_1. Round your answer for x_2 to three decimal places. [6 points]

Let $p(x) = x^3 - 5x^2 + 3$ polynomial $\Rightarrow$ cts f^n.

$p(0) = 3$ $p(1) = 1 - 5 + 3 = -1 < 0$

By OST (or IVT) p has a root $r \in [0,1]$.

$p(1)$ is closer to 0 than $p(0)$ $\Rightarrow$ $\boxed{x_1 = 1}$

Newton's Method: $x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$

$$x_{n+1} = x_n - \frac{[x_n^3 - 5x_n^2 + 3]}{[3x_n^2 - 10x_n]}$$

$$\therefore x_2 = 1 - \frac{[1-5+3]}{(3-10)} = 1 - \frac{[-1]}{[-7]} = 1 - \frac{1}{7}$$
$$= \frac{6}{7} = .\overline{857142}$$

$$\boxed{x_2 = .857}$$

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Some basic statistics (*)

N = 673 students wrote the test

$\overline{X} = 54.1\%$ = average

%	# of students	≈% of 673
100	0	0
90	21	3.1
80	79	11.7
70	70	10.4
60	93	13.8
50	126	18.7
40	101	15.0
30	85	12.6
20	59	8.8
10	24	3.6
1	15	2.2

# of students $\geq 50\%$	$389/676$ ≈	57.5
" " " $\geq 60\%$	$263/676$ ≈	39.1%
" " " $\geq 70\%$	$170/676$ ≈	25.3%
" " " $\geq 80\%$	$100/676$ ≈	14.8%
" " " $< 40\%$	$183/676$ ≈	27.2%
" " " $< 30\%$	$98/676$ ≈	14.6%

A surprising and disappointing number of students with scores $< 40\%$ and $< 30\%$. Very unusual for MATA32F and that the test was "standard + typical."

(*) All statistics calculated <u>after regrading</u> and before drop deadline.