* SOLUTIONS * * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32 - Midterm Test - Calculus for Management I

Examiners: R. Grinnell
G. Pete
B. Szegedy

Date: October 24, 2009
Time: 9:00 am
Duration: 110 minutes

Clearly indicate the following information:

Last Name (Print): ______________________

Given Name(s)(Print): ______________________

Student Number: * SOLUTIONS * * *

Signature: ______________________

Tutorial Number (e.g. TUT0032): ______________________

Carefully circle the name of your Teaching Assistant:

Jaehyun CHO	Amy JIANG	Amreen MOLEDINA
Fazle CHOWDHURY	Wenbin KONG	Sophia PENG
Duy Minh DANG	Kirill LEVIN	Sujanthan SRISKANDARAJAH
Paula EHLERS	Paul LI	Elena WANG
Xiaocong HAN	Xiao LIU	

Read these instructions:

1. This test has 11 pages. It is your responsibility to ensure that at the beginning of the test, all of these pages are included.
2. Put your solutions and/or rough work in the answer spaces provided. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are strongly encouraged to **write your test in pen or other ink**. Tests written in pencil will be denied any remarking or revision privilege.

* SOLUTIONS * * *

Print letters for the Multiple Choice Questions in these boxes:

1	2	3	4	5	6	7
b	b	d	b	c	a	a

Do not write anything in the boxes below.

Info.	**Part A**
3	21

Part B

1	2	3	4	5	6
12	12	14	14	14	10

Total
100

The following formulas may be helpful:

$$S = P(1+r)^n \qquad S = Pe^{rt}$$

$$S = R\left[\frac{(1+r)^n - 1}{r}\right]$$

$$A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

Part A - Multiple Choice Questions For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. If $f(x) = \dfrac{7x^3 + x}{6\sqrt{x}}$ then the value of $f'(1)$ is

   (a) 37/12 (b) 3 (c) 13/12 (d) 17/6 (e) none of (a) - (d).

$$f(x) = \frac{7}{6} x^{5/2} + \frac{1}{6} x^{1/2} \longrightarrow f'(x) = \frac{35}{12} x^{3/2} + \frac{1}{12} x^{-1/2}$$

$$f'(1) = \frac{35}{12} + \frac{1}{12} = \frac{36}{12} = 3$$

2. The future value of \$132.73 earning annual interest of 2% compounding continuously at the end of 246 months (to the nearest dollar, rounded up) is

   (a) \$198 (b) \$200 (c) \$202 (d) \$223 (e) none of (a) - (d).

$$FV = 132.73 e^{.41} \sim 200 \qquad \frac{(.02)(246)}{12} = .41$$

3. If $f(x) = \dfrac{x^2 - 3x - 18}{x^2 + 3x} + e^4 e^x$ then the value of $\lim\limits_{x \to -3} f(x)$ is

   (a) $3 + e^{-12}$ (b) $1 + e$ (c) $-3 + e$ (d) $3 + e$ (e) none of (a) - (d).

$$f(x) = \frac{(x+3)(x-6)}{x(x+3)} + e^4 e^x$$

$$= \frac{x-6}{x} + e^4 e^x \quad \forall x \neq -3$$

$$\therefore \lim_{x \to -3} f(x) = \frac{-9}{-3} + e^4 e^{-3} = 3 + e$$

4. If an investment increases by exactly 255% over 16 years under a constant periodic interest rate of r% compounding quarterly, then the annual interest rate is approximately

(a) 7.85% (b) 8.00% (c) 32.96% (d) 5.89% (e) none of (a) - (d).

$$3.55P = P\left(1+\frac{r}{4}\right)^{64} \rightarrow \left[(3.55)^{1/64}-1\right]4 = r$$

$\therefore\ r \sim 8.00$

5. If $y = \sqrt{4x^2+3}$ then y' equals

(a) $4xy$ (b) $\dfrac{3x}{y}$ (c) $\dfrac{4x}{y}$ (d) $\dfrac{4x}{y^2}$ (e) none of (a) - (d).

$$y' = \frac{1}{2}(4x^2+3)^{-1/2}(8x)$$

$$= \frac{4x}{\sqrt{4x^2+3}} = \frac{4x}{y}$$

6. The premiums on an insurance policy are \$40 per month, payable at the beginning of each month. If the policy holder wishes to pay for three year's premiums in advance, how much (rounded up to the nearest dollar) should be paid, provided that the interest rate is 6% APR compounding monthly ?

(a) \$1,322 (b) \$1,315 (c) \$1,282 (d) \$1,355 (e) none of (a) - (d).

$$A = 40 + 40\left[\frac{1-(1.005)^{-35}}{.005}\right] \qquad r = \frac{.06}{12}$$

$\sim 1{,}321.41$

"beginning of each month" $\rightarrow$ annuity due

7. If the average cost to produce $q > 0$ number of units is $\bar{c} = \dfrac{2^q}{q}$ then the $\underbrace{\text{marginal cost}}_{MC}$ at a production level of $q = 4$ is

(a) about 11.09 (b) 16 (c) 4 (d) about 1.77 (e) none of (a) - (d).

$$c(q) = \bar{c}(q)\cdot q = 2^q$$

$$MC = c'(q) = 2^q \cdot \ln(2)$$

$$MC\Big|_{q=4} = c'(4) = 2^4 \cdot \ln(2) \sim 11.09$$

Part B - Full Solution Questions Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32.

1. A total debt of $1,000 due now, $4,000 due 2 years from now, and $6,000 due 5 years from now is to be repaid by three payments:

   □ = debt ○ = payment

   (1) the first payment is made now.

   (2) the second payment (which is 80% of the first) is made at the end of 30 months from now.

   (3) the third payment (which is 60% of the second) is made at the end of 4 years from now.

   Interest is 4% APR compounding semi-annually. Calculate the amount of each of the three payments. Round your final answers up to the nearest dollar. A money-time diagram and an equation of value are required for full points [12 points]

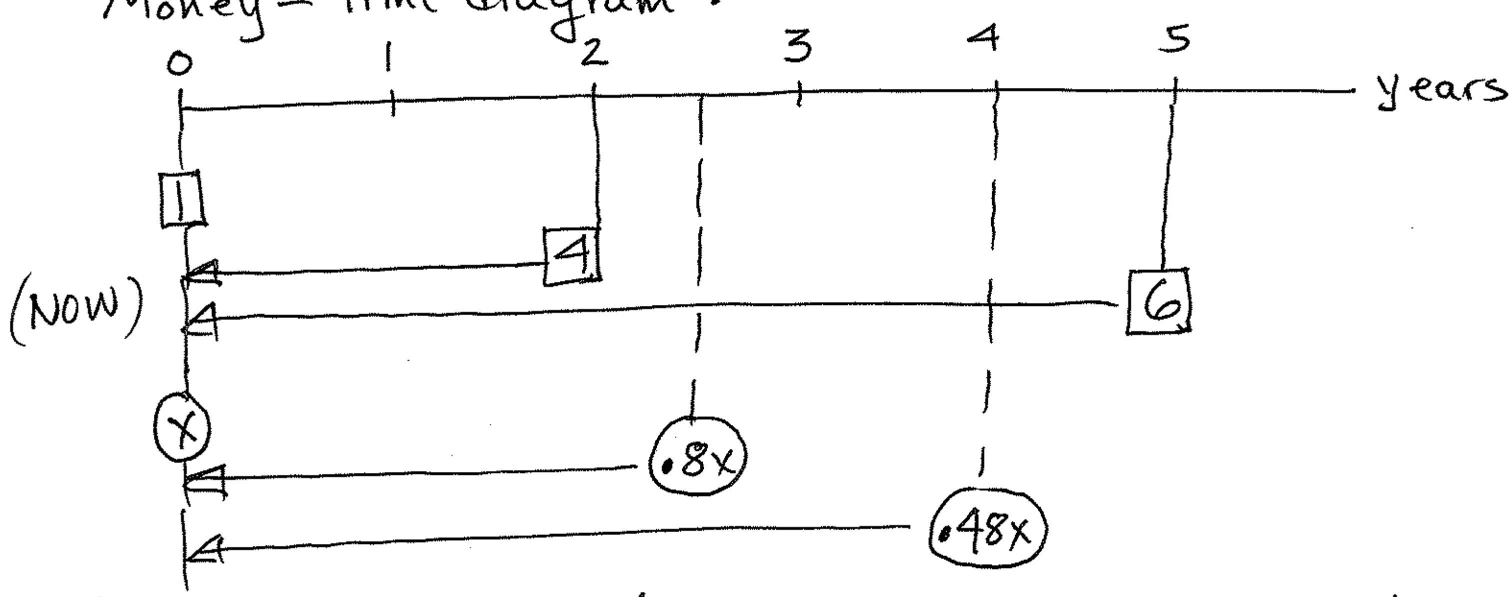
Let x = amount of 1st payment (in $1,000)

$\therefore$ $.8x$ = " " 2nd " (" ")

$.48x$ = " " 3rd " (" ")

All units are in $ 1,000. $r = \frac{.04}{2} = .02$

Money-Time diagram :



Equation of value (with calibration to time=0) :

$$x + (.8x)(1.02)^{-5} + (.48x)(1.02)^{-8} = 1 + 4(1.02)^{-4} + 6(1.02)^{-10}$$

$$2.13426002\!6x = 9.617471503 \text{ (approx)}$$

$$\therefore x = 4.506232 \text{ (approx)}$$

Answer :

1ST payment ~ $4,507
2nd payment ~ $3,605
3rd payment ~ $2,163

(Currency is $)

2. In all of this question assume $R > 0$ dollars is deposited into an ordinary annuity at the end of each quarter year. Interest is 5.2% APR compounding quarterly.

(a) Find the $\underbrace{\text{effective rate}}_{r_e}$ of interest expressed as a percentage (rounded up, to three decimal places). [3 points]

$$r_e = \left(1 + \frac{.052}{4}\right)^4 - 1 \sim 0.053022$$

Answer: $\boxed{r_e \sim 5.303\%}$

(b) If it is assumed that the annuity is empty to begin with, find the least number of years and quarter years it will take for the annuity to have a future value of $500R$. [9 points]

Interest compounds quarterly, so let n = # of "quarters" (i.e. # of compounding periods)

We solve

$$500R = R\left[\frac{(1.013)^n - 1}{.013}\right]$$

$$(1.013)^n = (500)(.013) + 1$$

$$n \ln(1.013) = \ln[7.5]$$

$$n = \frac{\ln[7.5]}{\ln(1.013)} \sim 155.99782$$

In order to actually reach the FV of $500R$, we round-up and take n as 156.

Answer: $\boxed{\text{39 years and 0 quarters}}$

3. (a) Differentiate and simplify: $y = 4x^2\sqrt{4x+1}$ [5 points]

$$y' = 8x\sqrt{4x+1} + \frac{4x^2(4)}{2\sqrt{4x+1}}$$

$$= \frac{8x}{\sqrt{4x+1}}(4x+1+x) = \boxed{\frac{8x(5x+1)}{\sqrt{4x+1}}}$$

(b) Find all point(s) (x, y) on the curve $y = f(x) = x^2e^{-3x}$ where the tangent line is horizontal. [6 points]

$$f'(x) = 2xe^{-3x} + x^2e^{-3x}(-3)$$

$$= xe^{-3x}(2-3x)$$

$$f'(x) = 0 \iff x = 0 \text{ or } x = \frac{2}{3}$$

$$f(0) = 0, \quad f\left(\frac{2}{3}\right) = \frac{4}{9}e^{-2}$$

Answer: Points are $(0,0)$ and $\left(\frac{2}{3}, \frac{4}{9}e^{-2}\right)$

(c) Assume $y = (1+c)u - ct$ where u is a function of the variable t and c is a positive constant. Show that $\frac{dy}{dt} = (1+c)\left[\frac{du}{dt} - \frac{c}{1+c}\right]$. [3 points]

$$\frac{dy}{dt} = (1+c)\frac{du}{dt} - c = (1+c)\left[\frac{du}{dt} - \frac{c}{1+c}\right]$$

4. In all of this question assume $r = f(q) = \dfrac{4q+6}{q+1} + 28q + 2$ is a total-revenue function (in dollars) for selling $q > 0$ units of a product.

(a) Find the $\underbrace{\text{marginal revenue}}_{MR}$ function and simplify it. [5 points]

$$MR = f'(q) = \frac{4(q+1) - (4q+6)(1)}{(q+1)^2} + 28$$

$$= \frac{4q+4-4q-6}{(q+1)^2} + 28$$

$$\boxed{MR = \frac{-2}{(q+1)^2} + 28}$$

(b) Find $\lim\limits_{q\to\infty}$ for the marginal revenue function and the average revenue function. [5 points]

$$\lim_{q\to\infty} MR$$

$$= \lim_{q\to\infty}\left[\frac{-2}{(q+1)^2} + 28\right]$$

$$= \boxed{28} \quad \left(\because \frac{-2}{(q+1)^2} \to 0 \text{ as } q\to\infty\right)$$

$$AR = \frac{f(q)}{q}$$

$$= \frac{4q+6}{q^2+q} + 28 + \frac{2}{q}$$

$$\therefore \lim_{q\to\infty} AR = \boxed{28}$$

$$\left(\frac{4q+6}{q^2+q},\ \frac{2}{q} \to 0 \text{ as } q\to\infty\right)$$

(c) Assume that for a certain "large" production level $q > 1000$, the revenue is \$70,006. Calculate a very good approximation to the revenue obtained by selling $q+1$ units. Justify your answer. [4 points]

Our assumption is that $f(q) = 70{,}006$ for some large $q > 1000$. From course material: $f(q+1) \sim f'(q) + f(q)$

$$\sim 28 + 70{,}006$$

$$= \underbrace{\boxed{70{,}034}}_{\text{Approx. revenue for selling } q+1 \text{ units.}}$$

($f'(q) \sim 28$ as from (b) above since q is 'large')

5. In all of this question let

$$f(x) = \begin{cases} (6-k)[(x+1)^{-1}] & if \quad x \geq 0 \\ k^2 e^x & if \quad x < 0 \end{cases}$$

where k is a real constant.

(a) The tangent line to the graph of $y = f(x)$ at the point where $x = 1$ has slope $m = 5$. Find the equation of this tangent line and write it in the form $y = mx + b$. [8 points]

We consider $f(x) = \dfrac{6-k}{x+1}$ as $1 \geqslant 0$.

$f'(x) = \dfrac{-(6-k)}{(x+1)^2}$

When $x=1$, $f'(1) = 5$,

so $5 = \dfrac{-(6-k)}{2^2}$

$\therefore k = 26$

Thus $f(1) = \dfrac{6-26}{2} = -10$

We have

$y + 10 = 5(x-1)$

so $\boxed{y = 5x - 15}$ is the desired equation.

(b) Find the value(s) of the constant k that make f continuous at 0. Sufficiently justify your solution. (Part(b) is independent of part(a)) [6 points]

To have continuity of f at 0 we require that $\lim\limits_{x \to 0} f(x) = f(0)$. This entails that

$\lim\limits_{x \to 0^-} f(x) = f(0)$. From the question,

$f(0) = 6-k$ and $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} k^2 e^x = k^2$

$\therefore$ solve $6-k=k^2$. We have $k^2+k-6=0$

so $(k+3)(k-2)=0$.

Answer: $\boxed{k = -3 \text{ or } 2}$

6. (a) Find $\lim_{x\to 1} \frac{xln(x)+2x-2}{x-1}$ (Warning: if you happen to know "l'Hopital's" rule, do not use it to find this limit) [5 points]

Let $f(x) = x\ln(x) + 2x$ so $f(1) = 2$

We observe that

$$\lim_{x\to 1} \frac{x\ln(x)+2x-2}{x-1} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$$= f'(1)$$

$$= \boxed{3}$$

$(f'(x) = \ln(x) + 1 + 2)$

(b) Let f and g be differentiable functions. Assume that for all real x, $g(f(x)) = x$ and $g'(x) = 1 + [g(x)]^2$. Find $f'(0)$. [5 points]

$$\frac{d}{dx}\left[g(f(x))\right] = \frac{d}{dx}(x) \rightarrow g'(f(x))\cdot f'(x) = 1$$

For $x=0$ we get $g'(f(0))\cdot f'(0) = 1$

But $g'(f(0)) = 1 + [g(f(0))]^2 = 1 + [0]^2 = 1$

$\therefore\ 1\cdot f'(0) = 1$ so $\boxed{f'(0) = 1}$