SOLUTIONS

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA32 - Calculus for Management I

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Date: October 29, 2007
Duration: 110 minutes

Clearly indicate the following information:

Family Name: SOLUTIONS

Given Name(s): __________

Student Number: __________

Signature: __________

Tutorial Number (e.g. TUT0032): __________

Carefully circle the name of your Teaching Assistant:

Marc CASSAGNOL	Carmen KU	Alexander WONG
Paula EHLERS	Jack LIN	Calvin WONG
Xiaocong HAN	Chris LIU	Yichao ZHANG
Mohammed KOBROSLI	Amreen MOLEDINA	Xiangqun ZOU
Wenbin KONG	Molu SHI	

Read these instructions:

1. This midterm test has 11 numbered pages. It is your responsibility to ensure that, at the beginning of the test, all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page or the last page. Indicate clearly the location of your continuing work.
3. With the exception of the Multiple Choice questions, full points are awarded only for solutions that are correct, complete and sufficiently display concepts and methods of MATA32.
4. You may use one standard hand-held calculator. The following devices are forbidden: laptop computers, Blackberry or similar devices, cell-phones, I-Pods, MP-3 players or similar devices.
5. Extra paper, notes and textbooks are forbidden.

SOLUTIONS

Do not write in the boxes below.

Info.	Part A
2	28

Part B

1	2	3	4	5	6	7	8
13	8	5	12	8	6	12	6

Total
100

The following formulas may be helpful:

$$S = P(1+r)^n$$

$$S = R\left[\frac{(1+r)^n - 1}{r}\right]$$

$$A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

Part A - Multiple Choice For each of the following, **circle the letter** next to the answer you think is most correct. Each correct answer earns 3.5 points and no answer/incorrect answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. The present value of \$ 5,000 due in four years at 6.6 % APR compounding monthly is about

   (a) \$ 4,891.50 (b) \$ 3,782.05 (c) \$ 3,935.49 (d) \$ 3,842.65

$$5{,}000(1.0055)^{-48} \sim 3{,}842.65$$

2. If $f(x) = \ln\left(e^{2x} + x\right)$ then the value of $f'(0)$ is

   (a) undefined (b) 3 (c) 2 (d) none of (a), (b), or (c).

$$f'(x) = \frac{2e^{2x}+1}{e^{2x}+x} \Rightarrow f'(0) = \frac{2+1}{1} = 3$$

3. Assume your annual salary increases from \$ 40,000 to \$ 60,000 over a four year period and that each annual increase occurs at the end of each of the four years. The annual rate of your salary increase is approximately

   (a) 10.67 % (b) 5 % (c) 12.5 % (d) none of (a), (b) or (c).

$$60 = 40(1+r)^4 \Rightarrow r = (1.5)^{1/4} - 1$$
$$\sim .1066$$

4. If $h(t) = \dfrac{t^3+3t^2+2t}{t^2+t-2} - \dfrac{1}{3}\ln(5+2t)$ then the value of $\lim_{t \to -2} h(t)$ is

   (a) 0 (b) -1 (c) $-\frac{2}{3}$ (d) undefined

Note that $\dfrac{(-2)^3+3(-2)^2+2(-2)}{(-2)^2+(-2)-2} = \dfrac{0}{0}$ .... a "$\frac{0}{0}$ - form"

$\therefore$ factor

$$h(t) = \frac{t\cancel{(t+2)}(t+1)}{\cancel{(t+2)}(t-1)} - \frac{1}{3}\ln(5+2t)$$

$$\therefore \lim_{t \to -2} h(t) = \frac{(-2)(-1)}{-3} - \frac{1}{3}\ln(1)$$
$$= -\frac{2}{3}$$

5. If $f(2) = 2$, $f'(2) = 3$, $g(2) = 2$ and $g'(2) = -5$ then

(a) $(g-f)'(2) = -2$ (b) $\left(\frac{f}{g}\right)'(2) = -1$ (c) $(fg)'(2) = -15$ (d) $(g \circ f)'(2) = -15$

X X X

$$(g \circ f)'(2) = g'(f(2)) \cdot f'(2)$$
$$= g'(2) \cdot f'(2) = (-5)(3) = -15$$

6. If the average cost to produce q units is given by $\bar{c} = \frac{600}{q+5}$ then the marginal cost at a production level of 5 units is

(a) 60 (b) 30 (c) -6 (d) 300

Cost $c = \bar{c} \cdot q = \frac{600q}{q+5}$

$$c' = \frac{600(q+5) - 600q}{(q+5)^2} \Rightarrow c'(5) = \frac{6000 - 3000}{(10)^2}$$
$$= 30$$

7. To three decimals, what approximate annual rate of interest compounding continuously is equivalent to 5.6 % APR compounding quarterly?

(a) 5.483 % (b) 5.644 % (c) 5.561 % (d) none of (a), (b) or (c).

$$e^r = \left(1 + \frac{.056}{4}\right)^4 \Rightarrow r = 4 \ln\left(1 + \frac{.056}{4}\right)$$
$$\sim .05561$$

8. The x-intercept of the tangent line to the curve $y = 2x + \frac{1}{\sqrt{x}}$ at the point where $x = 1$ is

(a) non-existent because the tangent line is horizontal (b) $-\frac{1}{5}$ (c) 1 (d) -1

$y = 2x + x^{-1/2} \rightarrow y' = 2 - \frac{1}{2} x^{-3/2}$

$y(1) = 3 \quad y'(1) = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ Tangent line eq^{n} is $y - 3 = \frac{3}{2}(x-1)$

When $y=0$, we have $x = (-3)\left(\frac{2}{3}\right) + 1 = -1$

Part B - Full Solution Problem Solving

1. (a) Find the exact value of $f'(2)$ where $f(x) = (2x+1)(6-5x)(x+9)+2^x$ [6 points]

$$f'(x) = 2(6-5x)(x+9) + (2x+1)(-5)(x+9) + (2x+1)(6-5x)(1) + 2^x \cdot \ln(2)$$

$$\therefore f'(2) = (2)(-4)(11) + (5)(-5)(11) + (5)(-4)(1) + 2^2 \ln(2)$$

$$= \boxed{-383 + \ln(16)}$$ .... exact value. (no decimals).

(b) Calculate the exact x-coordinate of the point(s) on the curve $y = (x^2+3x)e^{-2x}$ where the tangent line is horizontal. [7 points]

$$y' = (2x+3)e^{-2x} + (x^2+3x)(-2)e^{-2x}$$

Solve $y'=0$ and get

$$0 = e^{-2x}[(2x+3) + (x^2+3x)(-2)]$$

$$= e^{-2x}[-2x^2-4x+3]$$

$\because e^{-2x} > 0 \Rightarrow -2x^2-4x+3=0$

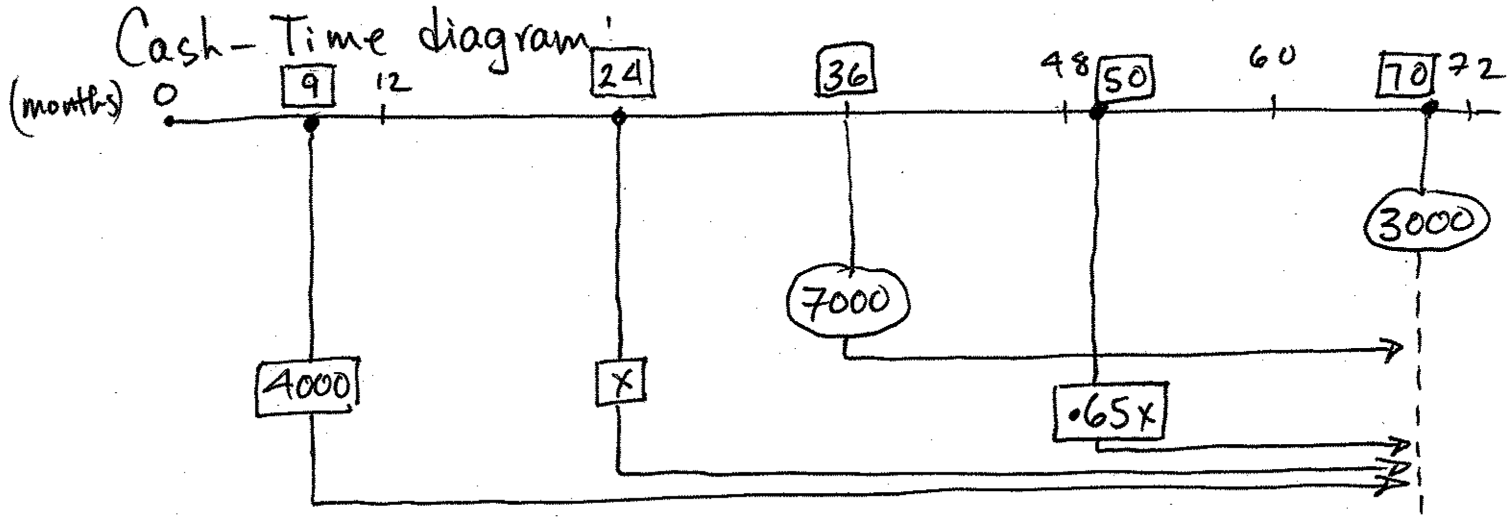
$$\therefore x = \frac{4 \pm \sqrt{16-4(-2)(3)}}{-4} = \frac{4 \pm \sqrt{40}}{-4}$$

$$= \boxed{\frac{-2 \pm \sqrt{10}}{2}}$$ are the exact x-coordinates where the tangent line is horizontal.

2. A total debt of \$ 7,000 due 3 years from now and \$ 3,000 due 70 months from now is to be repaid by three payments. The first payment is \$ 4,000 at the end of 9 months from now. The second payment is made at the end of 2 years from now and the third payment (which is 65 % of the second) is made at the end of 50 months from now. If interest is 6 % APR compounding monthly, how much are the second and third payments? (Round your final answers up to the nearest dollar). [8 points]

debt $\boxed{\text{pay}}$ $r = \frac{.06}{12} = .005$

Cash-Time diagram:



Equation of value calibrated to 70 months:

$$4000(1.005)^{61} + x(1.005)^{46} + (.65x)(1.005)^{20} = 3000 + 7000(1.005)^{34}$$

$$1.97606x = 5,871.2425$$

$$\therefore x = 2,971.13$$

$$.65x = 1,931.27$$

$\boxed{2^{nd} \sim \$2,971 \quad 3^{rd} \sim \$1,931}$

3. When 320 calculators are made during one work shift, the average cost is 29.55 and the marginal cost is 27.33 (both in units of dollars per calculator). Estimate to the nearest cent the total cost to make 321 calculators during one work shift. [5 points]

C = cost, q = quantity

Main relationship: $C(321) - C(320) \sim C'(320)$

$$\therefore C(321) \sim C'(320) + C(320)$$

$$= 27.33 + (29.55)(320)$$

$$= \boxed{9,483.33}$$

4. Find the limit or, if it does not exist, briefly state why it does not exist. Use the ∞ or $-\infty$ symbols where appropriate. [4, 4, 4 points]

(a) $\lim\limits_{x \to -\infty} \left(\dfrac{5x - 4x^3 + 2}{x^3 + 8x^2} + \dfrac{x+1}{1-x} \right)$

by course theory/lectures

$$= \lim_{x \to -\infty} \left(-\frac{4x^3}{x^3} + \frac{x}{-x} \right)$$

$$= -4 - 1 = \boxed{-5}$$

(b) $\lim\limits_{x \to 1} \left(\dfrac{e^{2x^2} - e^2}{x - 1} \right)$ (Tough Problem!)

Let $f(x) = e^{2x^2}$

$\therefore f'(x) = (4x)e^{2x^2}$

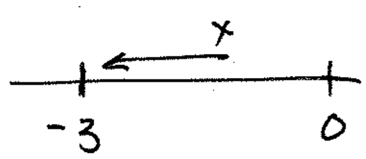
$\therefore f'(1) = 4e^2$

$$= \lim_{x \to 1} \left(\frac{f(x) - f(1)}{x - 1} \right)$$

$$= f'(1)$$

$$= \boxed{4e^2}$$

(c) $\lim\limits_{x \to -3^+} \left((1-x)^2 + \dfrac{x}{(x+3)^2} \right)$



Analysis: As $x \to -3^+$ $\quad x \sim -3$ and $-3 < x < 0$

$\dfrac{x}{(x+3)^2} \to -\infty$ and $(1-x)^2 \to 16$

$\boxed{\therefore \text{ limit above is } -\infty, \text{ so DNE}}$

5. For what value(s) of the constant c is the function f continuous on $(-\infty, \infty)$ where

$$f(x) = \begin{cases} cx+1 & if \quad x \leq 3 \\ cx^2 - 1 & if \quad x > 3 \end{cases}$$

Justify your solution completely. [8 points]

We analyze continuity for $x<3$, $x>3$, and at $x=3$.

For $x<3$, $f(x) = cx+1$ (polynomial), so cts.

For $x>3$, $f(x) = cx^2-1$ (polynomial), so cts.

$\therefore$ f is certainly continuous on $(-\infty, 3) \cup (3, \infty)$ for all real c.

For continuity at 3, we need

$$\lim_{x \to 3} f(x) = f(3) \quad (f(3) = 3c+1)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (cx+1) = 3c+1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (cx^2-1) = 9c-1$$

We need $3c+1 = 9c-1$

so $\boxed{c = \frac{1}{3}}$

6. Suppose you win a large amount of money in a lottery. There are two banks in which to deposit your good fortune. Bank A pays 5.4 % APR compounding semi-annually and Bank B pays $5\frac{1}{3}$ % APR compounding daily (1 year = 365 days). Which bank is the better choice to invest your winnings and why? [6 points]

We compare using effective rates.

Bank A: $r_e = \left(1 + \frac{.054}{2}\right)^2 - 1 \sim .054729$

Bank B: $r_e = \left(1 + \frac{.05\dot{3}}{365}\right)^{365} - 1 \sim .054777$

$\therefore$ We suggest investing with $\boxed{\text{Bank B}}$ as it has the higher effective rate.

7. In all of this question let $f(x) = \sqrt{4x^2+5}$ $= (4x^2+5)^{1/2}$

(a) Use the usual techniques of differentiation to find $f'(1)$. [5 points]

$$f'(x) = \frac{1}{2}(4x^2+5)^{-1/2} \cdot (8x)$$

$$\therefore f'(1) = \frac{8}{2(3)} = \boxed{\frac{4}{3}}$$

(b) Use the definition of derivative (i.e. first principles) to find $f'(1)$. [7 points]

$f(1) = 3$

$$f'(1) = \lim_{h\to 0}\left[\frac{f(1+h)-f(1)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sqrt{4(1+h)^2+5} - 3}{h}\right]$$

Multiplied top & bottom by $\sqrt{4(1+h)^2+5} + 3$

$$= \lim_{h\to 0}\frac{4(1+h)^2+5-9}{h(\sqrt{4(1+h)^2+5}+3)}$$

$$= \lim_{h\to 0}\frac{\cancel{4}+8h+4h^2+\cancel{5}-\cancel{9}}{h(\sqrt{4(1+h)^2+5}+3)}$$

$$= \lim_{h\to 0}\frac{8+4h}{\sqrt{4(1+h)^2+5}+3} = \frac{8}{3+3} = \boxed{\frac{4}{3}}$$

(Can also be obtained via

$$f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}\ )$$

8. Suppose \$ R is deposited into an ordinary annuity at the end of each month and that interest is 8.4 % APR compounding monthly. Let N represent the smallest whole number of years that it will take for the future value of the annuity to reach an amount of \$ $500R$.

Find the value of N. [6 points]

We solve for n where

$$500\cancel{R} = \cancel{R}\left[\frac{(1+.007)^n - 1}{.007}\right]$$

$$(1.007)^n = (500)(.007)+1$$

$$\therefore \quad n = \frac{\ln(4.5)}{\ln(1.007)} \sim 215.6194 \text{ months}$$

$\therefore$ n ~ 17.968 years

$\therefore$ $\boxed{N=18}$

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