MATA32F

Fall 2016 Midterm Test

Wed Oct 19 2016

SOLUTIONS

Part A

1	2	3	4	5	6	7
D	A	A	C	D	C	D

Part B Solutions appear on the pages to follow.

Simple statistics appear on the last page.

Part B Solutions

B1

(a) $y = \dfrac{2x+13}{4x+1}$ $\qquad \dfrac{dy}{dx} = \dfrac{2(4x+1) - (2x+13)(4)}{(4x+1)^2}$

$$= \frac{\cancel{8x}+2 - \cancel{8x} - 52}{(4x+1)^2}$$

$$= \frac{-50}{(4x+1)^2} \qquad \left.\frac{dy}{dx}\right|_{x=1} = \frac{-50}{25} = \boxed{-2}$$

(b) $h(t) = (3t+2)(5t-20)$

$$h'(t) = 3(5t-20) + (3t+2)(5)$$

$$= 15t - 60 + 15t + 10$$

$$= 30t - 50$$

$$0 \le h'(t) \le 3 \iff 0 \le 30t - 50 \le 3$$

$$\iff 50 \le 30t \le 53$$

$$\iff \boxed{\frac{5}{3} \le t \le \frac{53}{30}}$$

$\left(\text{or } t \in \left[\frac{5}{3}, \frac{53}{30}\right]\right)$ or

(c) $f(x) = \dfrac{x^2}{\ln(x)+1}$ $\qquad f'(x) = \dfrac{2x(\ln(x)+1) - x^2\left(\frac{1}{x}\right)}{(\ln(x)+1)^2}$

$$f'(e) = \frac{2e(2) - e}{2^2} = \frac{3e}{4} \qquad f(e) = \frac{e^2}{2}$$

Equation : $y - \dfrac{e^2}{2} = \dfrac{3e}{4}(x-e) = \dfrac{3e}{4}x - \dfrac{3e^2}{4}$

Slope-intercept form : $\boxed{y = \dfrac{3e}{4}x - \dfrac{e^2}{4}}$

B2

(a) Effective rate is $r_e = \left(1 + \frac{.048}{12}\right)^{12} - 1$

$$\approx .049070208$$

We take $\boxed{r_e \approx 4.907\%}$

(b) Let n = number of months. $r = \frac{.048}{12} = .004$

Solve $1000R = R\left[\frac{(1.004)^n - 1}{.004}\right]$

$$4 = (1.004)^n - 1$$

$$(1.004)^n = 5 \Rightarrow n \ln(1.004) = \ln(5)$$

$$n = \frac{\ln(5)}{\ln(1.004)} \approx 403.164$$

Round up to ensure that $\geq \$1{,}000R$ is actually reached. ∴ take 404 months.

404 months $= \boxed{33 \text{ years } \& \text{ } 8 \text{ months}}$

(c) Consider the PV of the annuity:

$$PV = R\left[\frac{1 - (1.004)^{-60}}{.004}\right] \approx 53.25R$$

Since $90R > PV$ it is better to take an amount of $\boxed{90R \text{ dollars now.}}$

B3

(a) $\lim_{x\to -5} \frac{x^2+25}{5-x} = \frac{(-5)^2+25}{5-(-5)} = \frac{50}{10} = \boxed{5}$

(b) $\lim_{x\to\infty} \left[\frac{3-5x+32x^3}{7+2x^2-x^3}\right]^{1/5} = \lim_{x\to\infty} \left[\frac{32x^3}{-1x^3}\right]^{1/5}$

$= (-32)^{1/5} = \boxed{-2}$

(c) Interpret $\lim_{x\to 0} \frac{\sqrt[3]{1+cx}-1}{x}$ as the definition of derivative.

Let $f(x) = \sqrt[3]{1+cx}$ $f(0) = 1$

$f'(x) = \frac{1}{3}(1+cx)^{-2/3}\, c = \frac{c}{3(1+cx)^{2/3}}$

$f'(0) = \frac{c}{3}$

We summarize:

$\lim_{x\to 0} \frac{\sqrt[3]{1+cx}-1}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$= f'(0) = \boxed{\frac{c}{3}}$

[One could also solve by using the difference of cubes formula and algebra]

Here is another solution to B3(c).

Observe that $(a-b)(a^2+ab+b^2) = a^3 - b^3$

$$\therefore \lim_{x \to 0}\left[\frac{\sqrt[3]{1+cx}-1}{x}\right]$$

$$= \lim_{x \to 0}\left[\frac{((1+cx)^{1/3}-1)}{x}\frac{((1+cx)^{2/3}+(1+cx)^{1/3}+1)}{((1+cx)^{2/3}+(1+cx)^{1/3}+1)}\right]$$

$$= \lim_{x \to 0}\left[\frac{1+cx-1}{x((1+cx)^{2/3}+(1+cx)^{1/3}+1)}\right]$$

$$= \lim_{x \to 0}\left[\frac{c}{(1+cx)^{2/3}+(1+cx)^{1/3}+1}\right] = \frac{c}{3}$$

B4

D_1 = debt #1 = $3{,}000(1.01)^{10}$ [30 months = 10 quarter years]

D_2 = debt #2 = $7{,}000(1.015)^{10}$ [5 years = 10 half years]

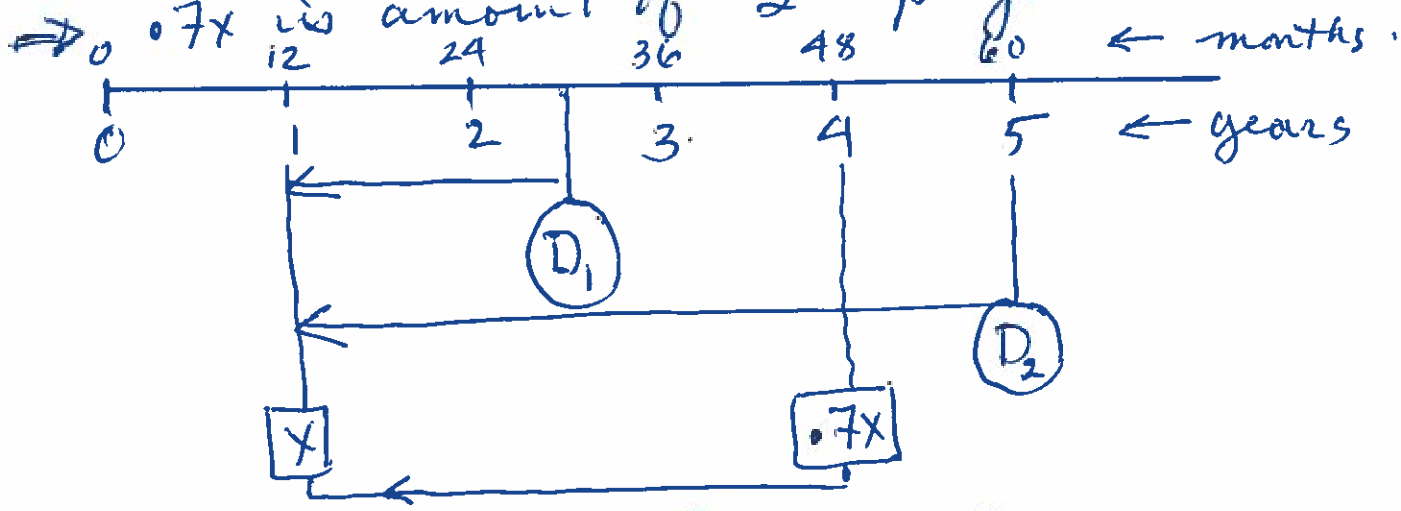
[r_1 = interest for $D_1 = \frac{.04}{4} = .01$ ✓
r_2 = interest for $D_2 = \frac{.03}{2} = .015$ ✓]

Global interest is $\frac{.036}{12} = .003$

Debt

$\boxed{\text{Pay}}$

Let x be amount of 1st payment
⇒ .7x is amount of 2nd payment.



Calibrate to time @ end of 1 year

$\boxed{\text{Payment Value}}$ Debt Value

$$x + (.7x)(1.003)^{-36} = 3000(1.01)^{10}(1.003)^{-18} + 7000(1.015)^{10}(1.003)^{-48}$$

$$1.6284409\,x = 3{,}139.917217 + 7{,}035.803108$$

$$x = \frac{10{,}175.7032}{1.6284409} \approx 6248.750$$

Rounded up to nearest dollar

$\boxed{\begin{array}{l} 1^{st}\text{ payment} = \$6{,}249 \\ 2^{nd}\text{ payment } \$4{,}375 \end{array}}$

B5

(a) Elasticity $\eta = \dfrac{p/q}{\frac{dp}{dq}} = \dfrac{\frac{1200-q^2}{q}}{-2q} = \boxed{\dfrac{1200-q^2}{-2q^2}}$

(b) Solve for q where $|\eta| = 1$

$|\eta| = \dfrac{1200-q^2}{2q^2} = 1$

$\Leftrightarrow 1200 - q^2 = 2q^2$

$1200 = 3q^2$

$q^2 = 400$ $\quad \boxed{\therefore q = 20}$ $(\because q > 0)$.

(c) Revenue $r = pq = (1200-q^2)q$

$= 1200q - q^3$

Marginal revenue $\dfrac{dr}{dq} = 1200 - 3q^2$

$\left.\dfrac{dr}{dq}\right|_{q=10} = 1200 - 300 = \boxed{900}$

B6

(a) Let the function be $f(x) = ax^2 + bx + c$ where a, b, c are constants.

Tangent is @ the point $(1, -2)$ so

$-2 = f(1) = a + b + c \leftarrow (1)$

Tangent is horizontal @ $(1, -2)$ so

$f'(x) = 2ax + b$ and $0 = f'(1) = 2a + b$

$\therefore\ b = -2a \leftarrow (2)$

When $x = 3$, $y = f(x)$ and $y = 3x + 1$ cross

$\therefore\ f(3) = 3(3) + 1 = 10$

$9a + 3b + c = 10 \leftarrow (3)$

Sub-in (1) to get $9a + 3b - 2 - a - b = 10$

$8a + 2b = 12$

Sub in (2) to get $8a + 2(-2a) = 12$

$4a = 12 \Rightarrow \boxed{a = 3}$ $\boxed{b = -6}$

$\boxed{c = -2 - 3 + 6 = 1}$

$\boxed{\therefore\ f(x) = 3x^2 - 6x + 1}$

(b) $$\left.\frac{dy}{dt}\right|_{t=3} = (h \circ g)'(3) = h'(g(3)) \cdot g'(3)$$

$$= h'(5) \cdot (-4)$$

$$= (-2)(-4) = \boxed{8}$$

Test Statistics

$N = 795$ students wrote the test

All statistics below are compiled <u>AFTER</u> regrading.

$\bar{x}$ = average $\approx 53.2\%$

	# of students	% of students
100%	1	$\approx .13$
90's	23	≈ 2.89
80's	66	≈ 8.30
70's	97	≈ 12.20
60's	109	≈ 13.71
50's	149	≈ 18.74
40's	127	≈ 15.97
30's	107	≈ 13.46
20's	78	≈ 9.81
10's	30	≈ 3.77
1's	8	≈ 1.01
	795	$\approx 100\%$

# of students with $\geq 50\%$ is 445 $\rightarrow \approx 56\%$

" " " $\geq 60\%$ is 296 $\rightarrow \approx 33\%$

" " " " $\geq 70\%$ is 187 $\rightarrow \approx 23.5\%$

" " " " $\geq 80\%$ is 90 $\rightarrow \approx 11.3\%$

" " " " $< 40\%$ is 223 $\rightarrow \approx 28\%$