* SOLUTIONS + SIMPLE STATISTICS *
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University of Toronto at Scarborough
Department of Computer and Mathematical Sciences
MATA32F (Calculus for Management I) Midterm Test

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Date: October 25, 2017
Duration: 110 minutes
Time: 5:00 pm

Last Name (PRINT) SOLUTIONS & STATISTICS -

First Name(s) (PRINT) (Page 11 →)

Student Number ______________________

Signature ______________________

Unambiguously circle your TA name and tutorial number

Bhakti BHATT 29

Bryan CHAN 10 14

Yuwei CHEN 1 11

Kayleigh (Soomin) CHOI 5 23

Yitian DING 24

Fan (Jiahao) FAN 2 12

Lei HUA 7

Yuanyuan LI 3 6

Mrigank MEHTA 27

Roleen NUNES 9 15

Dani (Daniella) O'CONNOR 16 22

Yang QIU 8 18 25

Menglu WANG 17 21

Binya XU 13

Annabelle (Can) ZHANG 4

Read these instructions:

1. This test has 11 numbered pages. Check that all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or page 11. Clearly indicate the location of your continuing work.
3. You may use <u>one</u> standard hand-held calculator (graphing is ok) that does not send/receive any electronic signals and does not communicates with any other electronic device. All other electronic devices (e.g. smart phones, smart watches, smart glasses, ear buds, music devices, etc.), extra paper, notes, textbooks, nontransparent pen/pencil carrying cases, and foods are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are encouraged to write your test in pen or other ink. If any portion of your test (i.e. Part A, Part B, any rough work) is written in pencil, then your entire test is denied any remarking privilege. There is no remarking of Part A regardless of whether you write in pen, ink, or pencil.

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
B	C	E	B	B	C	E

Do not write anything in the boxes below.

Info.	Part A
3	21

Part B

1	2	3	4	5	6
14	10	11	15	13	13

TOTAL
100

Some formulas:

$$S = P(1+r)^n \qquad S = Pe^{rt} \qquad \eta = \frac{p/q}{dp/dq}$$

Ordinary: $$S = R\left[\frac{(1+r)^n - 1}{r}\right] \quad \text{and} \quad A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

Due: $$S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R \quad \text{and} \quad A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$$

Part A (Multiple Choice Questions) For each of the following clearly print the letter of the answer you think is most correct in the boxes at the top of page 2. Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded. A small workspace is provided for your calculations.

1. If $f(x) = x\ln(x) + 2\sqrt{x}$ then $f'(1)$ equals

   (A) 0 (B) 2 (C) $1+e$ (D) $2+e$ (E) 3

$$f'(x) = \ln(x) + \frac{x}{x} + \frac{2}{2\sqrt{x}}$$

$$f'(1) = \ln(1) + 1 + 1 = 2$$

2. The value of $\lim\limits_{x\to 3} \dfrac{2x^3 - 2x^2 - 12x}{x^2 - 9}$ is

   (A) 5/2 (B) 11/6 (C) 5 (D) 4 (E) non-existent

$$\lim_{x\to 3} \frac{x(2x^2 - 2x - 12)}{(x-3)(x+3)}$$

$$= \lim_{x\to 3} \frac{2x(x^2 - x - 6)}{(x-3)(x+3)}$$

$$= \lim_{x\to 3} \frac{2x(x-3)(x+2)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{2x(x+2)}{x+3} = \frac{6(5)}{6} = 5$$

3. A child's education fund is set up by a single payment so that at the end of 12 years the fund has \$60,000 in it. Interest is 3.8% APR compounding semiannually. What is the approximate amount of the single payment to the nearest dollar rounded up?

   (A) \$24,515 (B) \$47,870 (C) \$40,183 (D) \$37,286 (E) \$38,192

$$r = \frac{3.8\%}{2} = .019$$

$$PV = 60{,}000\,(1.019)^{-24}$$

$$\approx 38{,}191.86$$

4. If $w = \dfrac{5}{2-x^2}$ then $\dfrac{dw}{dx}$ equals (A) $2w^2x$ (B) $\dfrac{2w^2x}{5}$ (C) $\dfrac{4w^2x}{5}$ (D) $-\dfrac{2w^2x}{5}$ (E) $\dfrac{5w^2x}{2}$ (F) none of (A) - (E)

$$\frac{dw}{dx} = 5\frac{(-1)}{(2-x^2)^2}(-2x) = 2x\left(\frac{5}{(2-x^2)^2}\right)$$
$$= \frac{2x}{5}\left(\frac{5}{2-x^2}\right)^2 = \frac{2}{5}x w^2$$

5. What principal invested now at 4% APR compounding quarterly will be worth approximately the same in five years as \$800 invested now at 5% compounding continuously?

(A) \$857 (B) \$842 (C) \$835 (D) \$826 (E) none of (A) - (D)

Solve for P in the equation

$$P(1.01)^{20} = 800e^{(.05)5}$$
$$P = 800(e^{.25})(1.01)^{-20} \approx 841.85$$

6. Let $c = f(q)$ be a cost function where q is quantity. Assume when $q = 4$, the average cost is 80 and the marginal cost is 48. What is the marginal average cost when $q = 4$?

(A) 12 (B) 32 (C) -8 (D) 18 (E) -16
(F) There is not enough information given to answer the question.

$$(\overline{C})'(q) = \left[\frac{C(q)}{q}\right]' = \frac{C'(q)\cdot q - C(q)}{q^2}$$

$C'(4) = 48$

$\overline{C}(4) = 80 = \dfrac{C(4)}{4}$

$$\therefore (\overline{C})'(4) = \frac{C'(4)\cdot 4 - C(4)}{16} = \frac{192-320}{16} = -8$$

$\therefore C(4) = 320$

7. If a is a positive constant, then the value of $\lim\limits_{x\to 0}\left[\dfrac{\sqrt{1+ax}-1}{ax}\right]$ is

(A) 0 (B) $2a$ (C) 1 (D) $-\frac{1}{2}$ (E) $\frac{1}{2}$ (F) $\frac{a}{2}$

$$\frac{1}{a}\cdot\lim_{x\to 0}\left(\frac{\sqrt{1+ax}-1}{x}\cdot\frac{\sqrt{1+ax}+1}{\sqrt{1+ax}+1}\right) = \frac{1}{a}\lim_{x\to 0}\left(\frac{1+ax-1}{x(\sqrt{1+ax}+1)}\right)$$
$$= \frac{1}{a}\cdot\lim_{x\to 0}\left(\frac{a}{\sqrt{1+ax}+1}\right) = \frac{1}{a}\cdot\frac{a}{2} = \frac{1}{2}$$

*** Make sure your answers are printed in the letter boxes at the top of page 2***

Part B (Full Solution Questions) Show all of your work. Answers/solutions will earn full points only if they are correct, complete, and display enough relevant concepts from MATA32F.

1. (a) Let $f(x) = \dfrac{e^{x^2}}{x}$. Find $f'(2)$ and leave your answer in terms of familiar mathematical constants, not decimals. [5 points]

$$f'(x) = \frac{(e^{x^2})(2x)(x) - (e^{x^2})(1)}{x^2}$$

$$= \frac{(e^{x^2})(2x^2) - e^{x^2}}{x^2}$$

$$\therefore f'(2) = \frac{(e^4)(8) - e^4}{4} = \frac{7}{4}e^4$$

(b) Find $\dfrac{du}{dt}$ where $u = t^3 \log_3(t) + 4\sqrt{t}$. [4 points]

$$\frac{du}{dt} = 3t^2 \log_3(t) + \frac{t^3}{[\ln(3)]t} + \frac{4}{2\sqrt{t}}$$

$$= 3t^2 \log_3(t) + \frac{t^2}{\ln(3)} + \frac{2}{\sqrt{t}}$$

(c) For $x > 0$, let $g(x) = \begin{cases} x+2 & if \quad x \geq 4 \\ k^2\sqrt{x} & if \quad x < 4 \end{cases}$ [5 points]

Find the value(s) of the constant k that makes g continuous at $x = 4$.

We require $\lim\limits_{x \to 4^-} g(x) = \lim\limits_{x \to 4^+} g(x) = g(4)$

$g(4) = 6$

$\lim\limits_{x \to 4^-} g(x) = \lim\limits_{x \to 4^-} k^2\sqrt{x} = 2k^2$

$\lim\limits_{x \to 4^+} g(x) = \lim\limits_{x \to 4^+} x+2 = 6$

$k^2 = 3$

$\therefore k = \pm\sqrt{3}$

2. A total debt of \$5,000 due at the end of four years from now and \$4,000 due at the end of seven years from now is to be repaid by three payments as follows:

(i) a first payment of \$6,000 now;

(ii) a second payment made at the end of three years from now;

(iii) a third payment (of an amount \$1,000 less than the second payment) is made at the end of six years from now.

Interest is 4% APR compounding semiannually. Find the payment in (ii) rounded-up to the nearest dollar. Then, based on this payment amount, find the payment in (iii). A complete money-time diagram and equation of value are required for full points. [10 points]

Let x represent amount of payment (ii).

$\therefore$ x − 1000 is the payment in (iii).

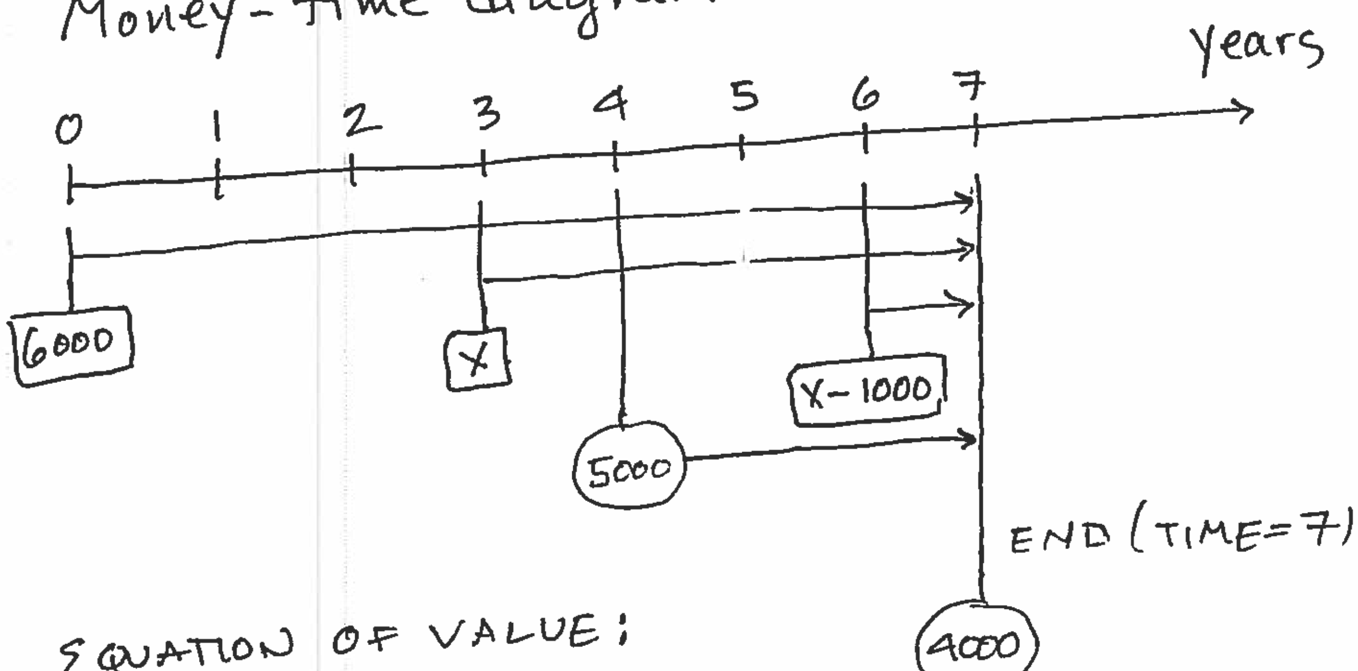
Periodic rate $= \frac{4\%}{2} = .02$

$\boxed{\text{pay}}$

debt

** CALIBRATE AT TIME = 7 **

Money-time diagram:



EQUATION OF VALUE:

$$6000(1.02)^{14} + x(1.02)^{8} + (x-1000)(1.02)^{2} = 5000(1.02)^{6} + 4000$$

$$\rightarrow x(1.02)^{8} + x(1.02)^{2} = 5000(1.02)^{6} + 4000 - 6000(1.02)^{14} + 1000(1.02)^{2}$$

$$2.212059x = 2,754.33952$$

$$x = 1,245.1474$$

Payment (ii)

\$1,246

Payment (iii)

\$246

3. The two parts of this question are independent of each other.

(a) Find $\lim_{x \to -\infty} \left[\frac{\sqrt{4x^2+1}}{x} - e^{1/x}\right]$. [4 points]

$$= \lim_{x \to -\infty} \left[\frac{\sqrt{x^2(4+\frac{1}{x^2})}}{x} - e^{1/x}\right] \qquad \left\{ \lim_{x \to -\infty} e^{1/x} = 1 \right.$$

$$= \lim_{x \to -\infty} \left[\frac{-x\sqrt{4+\frac{1}{x^2}}}{x} - e^{1/x}\right]$$

$$= \lim_{x \to -\infty} \left[-\sqrt{4+\frac{1}{x^2}} - e^{1/x}\right] = -2-1 = -3$$

(b) Find the value(s) of q for which the demand equation $p = 120 - 0.4q^2$ has unit elasticity. [7 points]

Elasticity of demand $\eta = \dfrac{p/q}{dp/dq}$ $\qquad \left\{ dp/dq = -.8q \right.$

$$\eta = \frac{\frac{120-0.4q^2}{q}}{-.8q} = \frac{120-0.4q^2}{-.8q^2}$$

Unit elasticity $\Leftrightarrow |\eta| = 1$

Solve $\dfrac{120-0.4q^2}{.8q^2} = 1$

$$120 - .4q^2 = .8q^2$$

$$120 = 1.2q^2 \longrightarrow q^2 = 100$$

$\because q \geq 0$, we only have unit elasticity at $q = 10$

4. In all of this question let $g(x) = \dfrac{10x}{x+1}$.

(a) Find $g'(x)$. [2 points]

$$g'(x) = \frac{10(x+1) - 10x}{(x+1)^2} = \frac{10}{(x+1)^2}$$

(b) Find all points (x, y) on the curve $y = g(x)$ for which the tangent line is parallel to the line $2x - 5y = 3$. [8 points]

Parallel lines have equal slopes.

$2x - 5y = 3$

$-5y = -2x + 3$

$y = \frac{2}{5}x - \frac{3}{5}$

$\therefore$ slope $= \frac{2}{5}$

Solve $\dfrac{10}{(x+1)^2} = \dfrac{2}{5}$ $\rightarrow$ $(x+1)^2 = 25$

$\therefore x = 4$ or -6

$g(4) = 8 \quad g(-6) = 12$

$\therefore$ points (x,y) are

$(4,8)$ and $(-6,12)$

(c) Calculate $\left.\dfrac{du}{dx}\right|_{x=0}$ where $u = g(f(x)) + f(0)(e^{g(x)})$ and f is a differentiable function such that $f(0) = 4$ and $f'(0) = 5$. [5 points]

Write $\dfrac{du}{dx}$ as $u'(x)$.

$$u'(x) = g'(f(x)) \cdot f'(x) + f(0) \cdot e^{g(x)} \cdot g'(x)$$

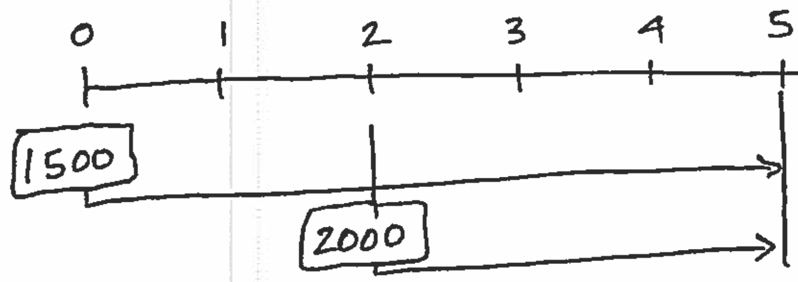
$$\therefore u'(0) = g'(f(0)) \cdot f'(0) + f(0) \cdot e^{g(0)} \cdot g'(0)$$

$$= g'(4) \cdot 5 + 4 \cdot 1 \cdot 10$$

$$= \frac{10}{25} \cdot 5 + 40 = 42$$

5. The two parts of this question are independent of each other.

(a) An amount of \$1,500 is invested now at 2% APR compounding continuously. Two years from now, a further amount of \$2,000 is invested at 3.6% APR compounding monthly. Find the sum total of both investments at the end of five years. Round the final sum total amount up to the nearest dollar. [5 points]



$$\frac{3.6\%}{12} = .003$$

END= 5years

$$SUM = 1500e^{(.02)(5)} + 2000(1.003)^{36}$$

$$= 1,657.756 + 2,227.735$$

$$= 3,885.491$$

∴ FINAL SUM is \$3,886

(b) A payment \$$R$ is invested at the beginning of each half year into an annuity due. Interest is 6.6% APR compounding semiannually. Calculate the smallest whole number of years for which the future value of the annuity due will reach the amount of \$200R. [8 points]

Let n be the number of compounding periods (each is 6 months long).

Periodic rate $r = \frac{6.6\%}{2} = 0.033$

Number of compounds and years must both be integers.

Solve for n where

$$200R = R\left[\frac{(1.033)^{n+1} - 1}{.033}\right] - R$$

$$200 = \frac{(1.033)^{n+1} - 1}{.033} - 1$$

$$\therefore (1.033)^{n+1} - 1 = (201)(.033) = 6.633$$

$$(1.033)^{n+1} = 7.633 \qquad n+1 = \frac{\ln(7.633)}{\ln(1.033)} \approx 62.601$$

$$\therefore n \approx 61.60$$

Must round-up to nearest integer

∴ use 62 compounds ⇒ 31 years

q = quantity, $C(q)$ = total cost

6. Assume the following throughout all of this question: a manufacturer finds that when 16 units of a product are made, the total cost of production is \$6,700 and the marginal cost is \$198.

(a) Find the approximate cost to produce 17 units. Round your answer up to the nearest dollar. [4 points]

$$C(17) \approx C(16) + C'(16)$$
$$= 6{,}700 + 198$$
$$= 6{,}898$$

In addition to the information above, assume that the total cost function C for the product has the form $C(q) = K\sqrt{q} + \frac{B}{q+4} + F$ where K, B, and F are positive constants and $q \geq 0$ is the quantity produced. Assume furthermore that the marginal cost for the first unit is \$768.

(b) Find the constants K, B, and F. [9 points]

We know that $C(16) = 6700$. ← ⊛

Marginal cost is $C'(q) = \frac{K}{2\sqrt{q}} - \frac{B}{(q+4)^2}$

Also, $\underbrace{C'(16) = 198}_{①}$ and $\underbrace{C'(1) = 768}_{②}$.

① → $\frac{K}{8} - \frac{B}{400} = 198$ ② → $\frac{K}{2} - \frac{B}{25} = 768$

$\therefore \frac{K}{2} - \frac{B}{100} = 792$ Combine by subtraction:

$$-\frac{B}{100} + \frac{B}{25} = 792 - 768 = 24$$

$\therefore \frac{3}{100}B = 24 \rightarrow B = 800$. Use ①: $\frac{K}{8} = 198 + 2$

$K = 1600$

Lastly, we know that $C(16) = 6700$ (by ⊛)

$\therefore 4K + \frac{B}{20} + F = 6700 \rightarrow F = 6700 - 4(1600) - \frac{800}{20}$
$= 260$

FINAL ANSWER: $K = 1{,}600$ $B = 800$ $F = 260$

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SIMPLE STATISTICS

All numbers, data, statistics below are taken after Re-grades & Drop course.

N = # who wrote test and did not drop course
= 636

Average $\bar{x} = 66.0\%$

% range	% of 636
100 →	0.16
90's →	11.6
80's →	13.9
70's →	18.9
60's →	20.8
50's →	14.4
40's →	11.0
30's →	4.5
20's →	2.7
10's →	1.6
1's →	0.5
	≈ 100 (small rounding errors)

% who passed (i.e. $\geq 50\%$) ≈ 79.8

% who got $\geq 60\%$ ≈ 65.4

% who got $\geq 70\%$ ≈ 44.6

% who got $\geq 80\%$ ≈ 25.7

Highest Mark = 100%

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