SOLUTIONS + Basic Stats

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32S - Midterm Test - Calculus for Management I

Examiner: R. Grinnell

Date: March 9, 2013
Duration: 120 minutes
Time: 1:00 pm

Last Name (PRINT): SOLUTIONS + BASIC STATS

First Name(s) (PRINT): ______________________

Student Number: ______________________

Signature: ______________________

Circle your TA and Tutorial Number:

Fazle CHOWDHURY 1 2 4 5

Qi ZHAO 6

Instructions:

1. This test has 11 numbered pages. It is your responsibility to ensure at the start of the test that all 11 pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator that does not perform any: graphing, matrix operations, numerical/symbolic differentiation or integration. If your calculator performs any of these operations, then you can not use it during the test. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are strongly encouraged to write your test in pen or other ink. If any questions are answered in pencil, then your entire test is denied any remarking privilege.
5. If you brought a cell/smart phone into the test room, it must be turned off and left at the front of the room.

SOLUTIONS + BASIC STATS

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
B	B	E	B	C	E	A

Do not write anything in the boxes below.

Info.	Part A
2	21

Part B

1	2	3	4	5	6
12	14	13	12	16	10

Total
100

Some formulas:

$$S = P(1+r)^n \qquad S = Pe^{rt}$$

revenue = price × quantity   profit = revenue - cost

Ordinary: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

Due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

$$\eta = \frac{p/q}{dp/dq}$$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. If $f(x) = \dfrac{6x^4 + 2x}{3\sqrt{x}}$ then the value of $f'(1)$ is

   (A) 8/3 (B) 22/3 (C) 17/6 (D) 11/3 (E) a number not in (A) - (D)

Simplify 1ST: $f(x) = 2x^{7/2} + \frac{2}{3}x^{1/2}$

$f'(x) = 7x^{5/2} + \frac{1}{3}x^{-1/2}$

$f'(1) = 7 + \frac{1}{3} = \boxed{\frac{22}{3}}$

2. The present value of \$200 due at the end of 246 months at an annual rate of 2% compounding continuously is (to the nearest cent, rounded down)

   (A) \$123.73 (B) \$132.73 (C) \$301.36 (D) \$142.67 (E) none of (A) - (D)

$PV = 200e^{-.41}$

$\approx \boxed{132.73}$

$t = \frac{246}{12} = 0.5$ years

$r = .02$

$\therefore rt = 0.41$

3. If $h(x) = \dfrac{-x^2 + x + 6}{x^2 + 2x} + e^5 e^x$ then $\lim_{x \to -2} h(x)$ equals

   (A) $\frac{5}{2} - e^3$ (B) $\frac{5}{2} + e^3$ (C) $-\frac{5}{3} + e^3$ (D) $\frac{5}{3} - e^2$ (E) $-\frac{5}{2} + e^3$

$\dfrac{-x^2+x+6}{x^2+2x} = \dfrac{-(x^2-x-6)}{x(x+2)} = \dfrac{-(x-3)(x+2)}{x(x+2)} = \dfrac{-(x-3)}{x}, \; x \neq -2$

$\therefore \lim_{x \to -2} h(x) = \lim_{x \to -2} \left[\dfrac{-(x-3)}{x} + e^5 e^x \right] = \dfrac{-(-2-3)}{-2} + e^5 e^{-2}$

$= \boxed{-\frac{5}{2} + e^3}$

4. If an investment increases by exactly 255% over 16 years under a constant periodic interest rate of r% compounding quarterly, then the annual interest rate is approximately

(A) 7.85% (B) 8.00% (C) 32.96% (D) 5.89% (E) none of (A) - (D)

Consider $3.55R = R(1+r)^{64}$ $\quad 64 = 4(16)$

$\therefore\ r = (3.55)^{1/64} - 1 \Rightarrow APR = 4r$
$= 4\left[(3.55)^{1/64} - 1\right] \approx \boxed{8.00\%}$

5. If a and b are positive real constants and $u = \sqrt{at^2 + b}$ then $\dfrac{du}{dt}$ equals

(A) $2atu$ (B) $\dfrac{2at}{u}$ (C) $\dfrac{at}{u}$ (D) $\dfrac{at}{2u}$ (E) none of (A) - (D)

$$\frac{du}{dt} = \frac{1}{2}(at^2+b)^{-1/2}(2at)$$
$$= \frac{at}{(at^2+b)^{1/2}} = \boxed{\frac{at}{u}}$$

6. The least whole number of months it takes an investment to increase by 23% at 2.4% APR compounding eight times annually is

(A) 98 (B) 69 (C) 70 (D) 104 (E) 105 (F) 106

8 compounds per year $\Rightarrow$ 1.5 months per compound.

Let $n = \#$ of compounds, $n \in \mathbb{N}$ $\quad r = \frac{2.4}{100} \times \frac{1}{8} = .003$

Consider $1.23P = P(1.003)^n$

$\Rightarrow \ln(1.23) = n\ln(1.003)$

$n = \dfrac{\ln(1.23)}{\ln(1.003)} \approx 69.108$

MUST ROUND UP!

$N = 70$ compounds

$\therefore (1.5)(70) = \boxed{105 \text{ months}}$

7. If the average cost to produce $q > 0$ number of units is $a = \dfrac{3^q}{4q}$ then the marginal cost at a production level of $q = 4$ is approximately

(A) 22.25 (B) 20.25 (C) 27 (D) 4.30 (E) none of (A) - (D)

Cost $= C = aq = \left(\dfrac{3^q}{4q}\right)q = \dfrac{3^q}{4}$

Marginal Cost $= \dfrac{dC}{dq} = \dfrac{3^q}{4} \cdot \ln(3)$

$\therefore \left.\dfrac{dC}{dq}\right|_{q=4} = \dfrac{3^4}{4} \cdot \ln(3)$

$\approx \boxed{22.2468}$

Make sure that your answers are printed in the letter boxes at the top of page 2

Part B (Full Solution Questions) Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32S.

1. A total debt of \$2,000 now and \$5,000 due 3 years from now and \$3,500 due 3 years after that is to be repaid by three payments as follows:

   (i) a first payment now;

   (ii) a second payment (which is three-quarters of the first payment) made at the end of 18 months from now;

   (iii) a third payment (which is 80% of the second payment) made five years from now.

   Interest is 3% APR compounding semi-annually. Find the amount of the the three payments and round your final answers up to the nearest dollar.

   (A correct, complete money-time diagram and equation of value are worth 2 points each.)

   [12 points]

All money is in units of \$1,000

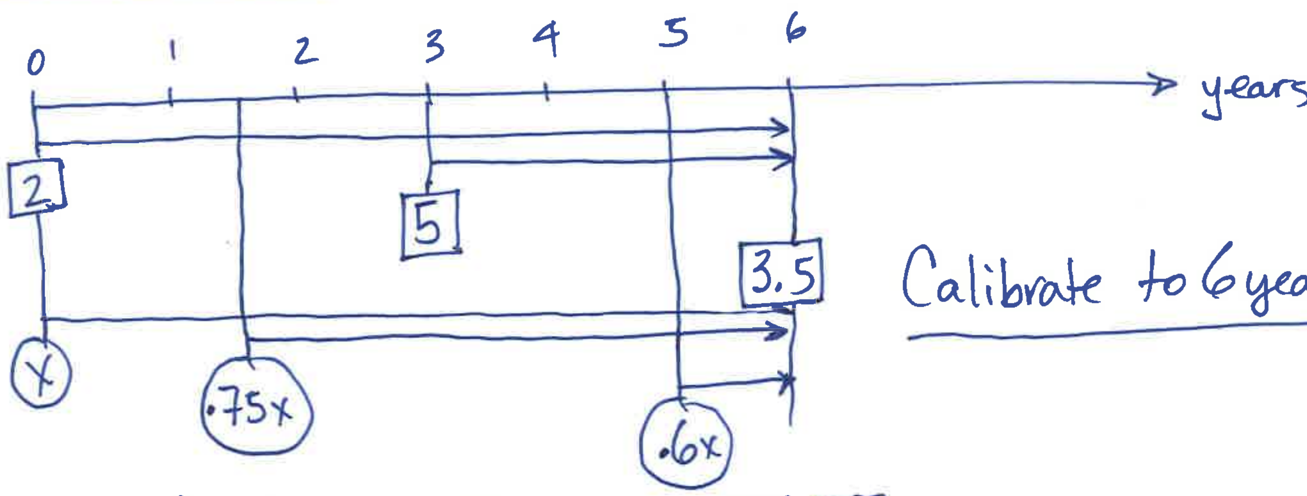
$r = \text{periodic rate} = \frac{3}{100} \times \frac{1}{2} = .015$

$x = \text{amount of } 1^{ST} \text{ payment}$

$\therefore\ 0.75x = \text{amount of } 2^{nd} \text{ payment}$

$(0.8)(0.75)x = 0.6x = \text{amount of } 3^{rd} \text{ payment.}$

Money-time diagram : $\square$: Debt $\bigcirc$: Pay



Calibrate to 6 years

Equation of value: $\sum \bigcirc = \sum \square$

$$x(1.015)^{12} + (.75x)(1.015)^9 + (.6x)(1.015)^2$$

$$= 3.5 + 5(1.015)^6 + 2(1.015)^{12}$$

We get $2.67129565\,3x = 11.35845266$

$\therefore\ x \approx 4.252038763$

$$\boxed{\begin{aligned} 1^{ST} &\approx 4,253 \\ 2^{nd} &\approx 3,190 \\ 3^{rd} &\approx 2,552 \end{aligned}}$$

2. The parts of this question are independent of each other.

(a) Differentiate the function $f(x) = 3x^2\sqrt{6x+5}$ and simplify your answer fully.

[5 points]

Re-write: $f(x) = 3x^2(6x+5)^{1/2}$

$\therefore f'(x) = 6x(6x+5)^{1/2} + 3x^2\left(\frac{1}{2}\right)(6x+5)^{-1/2}(6)$

$= (6x+5)^{-1/2}(3x)\left[2(6x+5) + 3x\right]$

$= \dfrac{(3x)(15x+10)}{\sqrt{6x+5}} = \boxed{\dfrac{(15x)(3x+2)}{\sqrt{6x+5}}}$

(b) Find the equation of every horizontal tangent line to the curve $y = g(x) = 4x^2e^{-2x}$.

[5 points]

Slope: $y' = 8xe^{-2x} + 4x^2e^{-2x}(-2)$

$= e^{-2x}(8x)\left[1 - x\right]$

Horizontal $\Rightarrow$ Solve $y' = 0$

We get $x = 0$ or $x = 1$

$g(0) = 0$, $g(1) = 4e^{-2}$

$\therefore$ equations of horizontal tangent lines are $y = 0$ and $y = \frac{4}{e^2}$

(c) Assume u is a function of the variable t and $w + \alpha t = u + \alpha u$ where α is a positive real constant. Show that $\dfrac{dw}{dt} = c\left[\dfrac{du}{dt} + \beta\right]$ where $c = 1 + \alpha$ and $\beta = -\dfrac{\alpha}{1+\alpha}$. [4 points]

$W = u + \alpha u - \alpha t$

$\therefore \dfrac{dw}{dt} = \dfrac{du}{dt} + \alpha\dfrac{du}{dt} - \alpha$

$= (1+\alpha)\dfrac{du}{dt} - \alpha$

$= (1+\alpha)\left[\dfrac{du}{dt} - \dfrac{\alpha}{1+\alpha}\right]$

$= c\left[\dfrac{du}{dt} + \beta\right]$

as required.

3. In all of this question, assume an interest rate of 5.1% APR that is compounding every third of a year.

(a) Find the $\overbrace{\text{effective rate}}^{r_e}$ of interest expressed as a percentage, rounded down to three decimals. [3 points]

$$r = \frac{5.1}{100} \times \frac{1}{3} = 0.017 \qquad \therefore \boxed{r_e \approx 5.187\%}$$

$$r_e = (1+r)^3 - 1 = (1.017)^3 - 1 \approx 5.1871913\%$$

(b) Find the smallest APR (expressed as a percentage, rounded down to three decimals) that is equivalent to the assumed interest rate given at the top of this question. [3 points]

Smallest APR $\Leftrightarrow$ compounding is <u>continuous</u>!

Let r represent the smallest APR

Consider $Pe^r = P(1.017)^3$

$\Rightarrow e^r = (1.017)^3 \rightarrow r = 3\ln(1.017)$

≈ 5.05713512

$\therefore \boxed{r \approx 5.057}$

(c) Assume an ordinary annuity is empty to begin with and that $R > 0$ dollars are deposited into it at the end of every third of a year. Find the least whole number of years it will take for the annuity to have a future value of $200R$. [7 points]

From above $r = 0.017$

Let $n = \#$ of compounds, $n \in \mathbb{N}$.

Consider $200R = R\left[\frac{(1.017)^n - 1}{.017}\right]$

3 compounds per year $= \frac{1}{3}$ year per compound.

$(200)(.017) = (1.017)^n - 1$

$\therefore (1.017)^n = (200)(0.017) + 1$

$$n = \frac{\ln[(200)(0.017)+1]}{\ln(1.017)}$$

≈ 87.8919

MUST ROUND UP TO NEAREST NATURAL NUMBER

$\therefore$ 88 compounds

$= 29\frac{1}{3}$ years

MUST ROUND YEARS UP!

$\boxed{\therefore \text{ 30 YEARS}}$

4. In all of this question, let $f(x) = \dfrac{x^{-2} - a^{-1}}{x^{-2} - a^{-2}}$ where $a > 2$ is a constant.

(a) Find all points of discontinuity of f. [3 points]

Re-write: $f(x) = \dfrac{\frac{1}{x^2} - \frac{1}{a}}{\frac{1}{x^2} - \frac{1}{a^2}}$

$\boxed{\therefore \text{ points of discontinuity are } x = 0, a, -a}$

f is discontinuous $\Leftrightarrow$ f is UNDEFINED

(b) Express f as a rational function. [3 points]

$$f(x) = \frac{\left(\frac{a-x^2}{ax^2}\right)}{\left(\frac{a^2-x^2}{a^2x^2}\right)} = \frac{a-x^2}{ax^2} \cdot \frac{a^2x^2}{a^2-x^2} = \frac{(a-x^2)a}{a^2-x^2} = \frac{a^2-ax^2}{a^2-x^2}$$

$$= \frac{ax^2+a^2}{-x^2+a^2} = \boxed{\frac{ax^2-a^2}{x^2-a^2}}$$

(c) Find $f'(2)$ and simplify your answer by factoring completely. [6 points]

$$f'(x) = \frac{(2ax)(x^2-a^2) - (ax^2-a^2)(2x)}{(x^2-a^2)^2}$$

$$\therefore f'(2) = \frac{(4a)(4-a^2) - (4a-a^2)(4)}{(4-a^2)^2}$$

$$= \frac{16a - 4a^3 - 16a + 4a^2}{(4-a^2)^2}$$

$$= \boxed{\frac{-4a^2(a-1)}{(4-a^2)^2}}$$

5. In all of this question let $q > 0$ represent the number of units of a product (i.e. quantity) sold and let $p = \dfrac{585}{2q+6}$ be the unit price of the product when q units are sold (p is also called the demand function). Assume that q can be any positive real quantity.

(a) Find the marginal revenue when $q = 2$. $r = pq$ [5 points]

$r = \dfrac{585q}{2q+6}$ Marginal revenue $= \dfrac{dr}{dq}$

$$\frac{dr}{dq} = \frac{585(2q+6) - 585q(2)}{(2q+6)^2} = \frac{6(585)}{(2q+6)^2}$$

$$\therefore \left.\frac{dr}{dq}\right|_{q=2} = \frac{6(585)}{100} = \boxed{35.1}$$

(b) Let $c = f(q)$ be a cost function. Assume the marginal cost of 7 units is 6.3 and the average cost of 7 units is 12. Estimate the profit obtained when 8 units are sold.

[5 points]

$P =$ profit $= r - c$

$r(8) = \dfrac{(585)(8)}{22} = 212.\overline{72}$

$a(7) = 12 \rightarrow C(7) = 84$

$$P(8) = r(8) - c(8)$$
$$\approx r(8) - [c(7) + c'(7)]$$
$$= 212.\overline{72} - [84 + 6.3]$$
$$\approx 122.43$$

$\boxed{\therefore \text{ Profit is about } 122.43}$

(c) Show that demand is elastic for all $q > 0$. [6 points]

$\eta = \dfrac{\frac{p}{q}}{p'}$ $\quad p' = \dfrac{(-585)(2)}{(2q+6)^2} = \dfrac{-1170}{(2q+6)^2}$

$$\therefore \eta = \frac{\left(\frac{585}{q(2q+6)}\right)}{\left(\frac{-1170}{(2q+6)^2}\right)} = \frac{585}{q(2q+6)} \cdot \frac{(2q+6)^2}{-1170} = -\frac{1}{2}\,\frac{2q+6}{q}$$

$$|\eta| = \frac{1}{2}\,\frac{2q+6}{q} = \frac{1}{2}\left(2 + \frac{6}{q}\right) > \frac{1}{2}(2) = 1 \text{ for all } q > 0$$

$\because |\eta| > 1$ for all $q > 0$ $\therefore$ demand is elastic for all $q > 0$.

6. In all of this question, let f be the function defined for $x > 0$ by $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ if $0 < x < 4$ and $f(x) = 2k^2 - kx$ if $x \geq 4$ where k is a positive real constant.

(a) Find the value(s) of k for which f is continuous at $x = 4$. Sufficiently justify your solution. [7 points]

f is continuous at 4 $\Longleftrightarrow \underbrace{\lim_{x\to 4} f(x) = f(4)}_{(*)}$

$f(4) = 2k^2 - 4k$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = 5/2$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\left(2k^2 - kx\right) = 2k^2 - 4k$

By $(*)$: $2k^2 - 4k = \frac{5}{2}$

$\Rightarrow 4k^2 - 8k - 5 = 0$

$k = \frac{8 \pm \sqrt{64 + 80}}{8} = \frac{8 \pm 12}{8} = 2.5$ and $\underbrace{-.5}_{\text{Want } k>0}$

$\boxed{\because k > 0 \text{ the only value of } k \text{ is } k = 2.5}$

(b) Find and simplify the function $F(x) = \frac{x-1}{2x\sqrt{x}} - f'(x)$ where $0 < x < 4$. [3 points]

When $0 < x < 4$, $f(x) = x^{1/2} + x^{-1/2}$

$\therefore f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

$\therefore F(x) = \frac{x-1}{2x\sqrt{x}} - \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)$

$= \frac{x - 1 - x + 1}{2x\sqrt{x}} = 0$

$\boxed{\therefore F(x) = 0 \text{ for all } x \in (0,4)}$

Basic Statistics (*)

~~(This page is intentionally left blank)~~

$N = 134$ = # of students who wrote the test

$\bar{X}$ = average $\approx 49.3\%$

# of students passing ($\geq 50\%$) $= \frac{68}{134} \approx 50.7\%$

of " $\geq 60\% = \frac{36}{134} \approx 26.9\%$

of " $\geq 70\% = \frac{12}{134} \approx 9.0\%$

	# of students	$\approx$ % of 134
90's	0	0
80's	3	2.2
70's	9	6.7
60's	24	17.9
50's	32	23.9
40's	31	23.1
30's	21	15.7
20's	10	7.5
10's	1	0.7
1's	3	2.2

(*) All statistics are calculated before regrading