SOLUTIONS + Basic Stats

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32F (Calculus for Management I) Midterm Test

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Date: November 1, 2013
Duration: 110 minutes
Time: 7:00 pm

Last Name (PRINT): Solutions

First Name(s) (PRINT): ____________________

Student Number: ____________________

Signature: ____________________

Circle your TA name and tutorial number:

Rui GAO	8 10	(Ethan) Junsheng WU	1 3
Anran JIA	7 16	Kevin YAN	2
(Reggie) Zejun LIU	4 5 17	(Philip) Jianhao YANG	13 15
Elina MAN	22 23	(Ric) Biyun ZHANG	9 12
(Jerry) Chao SHEN	6 21	(James) Yang ZHOU	14 18
Huiyi WANG	20 24	Manci ZHU	19

Read these instructions:

1. This test has 11 numbered pages. At the start of the test check that all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or page 11. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator that does not perform any: graphing, matrix operations, numerical/symbolic differentiation or integration. If your calculator performs any of these operations, then you can not use it during the test. All other electronic devices, extra paper, notes, textbooks, pen/pencil carrying cases, and foods are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are encouraged to write your test in pen or other ink. If any questions are answered in pencil, then your entire test is denied any remarking privilege.

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
D	E	A	B	D	A	D

Do not write anything in the boxes below.

2 points if and only if all info is included on page 1. 2

Info.	Part A
2	21

Part B

1	2	3	4	5	6
13	11	12	12	15	14

Total
100

Some formulas:

$$S = P(1+r)^n \qquad S = Pe^{rt} \qquad \eta = \frac{p/q}{p'}$$

Ordinary: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

Due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded. A small workspace is provided for your calculations.

1. If a is a positive constant and $f(x) = \dfrac{12x^3 + 2ax}{\sqrt{x}}$ then $f'(1)$ equals

(A) $72 + 4a$ (B) $48 + 3a$ (C) $30 + \frac{a}{2}$ $\boxed{\text{(D) } 30 + a}$ (E) $32 + a$

Simplify 1ST : $f(x) = 12x^{5/2} + 2ax^{1/2}$

$f'(x) = 30x^{3/2} + ax^{-1/2}$

$\boxed{f'(1) = 30 + a}$

2. If $\bar{c} = 0.03q + 12 + 200\dfrac{ln(q)}{q}$ is an average cost function, then the marginal cost when $q = 100$ is

(A) 20.922 (B) 15.047 (C) 0.020 (D) 15.000 $\boxed{\text{(E) } 20.000}$

C = cost $\bar{c} = \frac{C}{q} \Rightarrow C = .03q^2 + 12q + 200\ln(q)$

$MC = c' = .06q + 12 + \frac{200}{q}$

$c'(100) = 6 + 12 + 2 = \boxed{20}$

3. The least whole number of months it takes an investment to increase by 22% at a periodic rate of 0.94% compounding at the end of every three months is

$\boxed{\text{(A) } 66}$ (B) 67 (C) 63 (D) 84 (E) 81 (F) none of (A) - (E)

$r = \frac{0.94}{100} = .0094$ Let n = # of 3-month compoundings

Solve $1.22 = (1.0094)^n$

$\ln(1.22) = n\ln(1.0094)$

$n = \dfrac{\ln(1.22)}{\ln(1.0094)} \approx 21.2536$

$\therefore$ we take 22 of the 3-month compoundings. $\therefore$ Least whole number of months is $22 \times 3 = \boxed{66}$

4. If $F(x) = \dfrac{x+2x^3}{4x^3+4x} + e^{1/x} + \sqrt{3 + \dfrac{x+1}{x}}$ then $\lim\limits_{x\to\infty} F(x)$ equals

(A) $\frac{3}{2}$ $\boxed{\text{(B) } \frac{7}{2}}$ (C) $\frac{13}{4}$ (D) $\frac{13}{2}$ (E) a number not in (A) - (D)

(F) no number, because the limit does not exist

$$\lim_{x\to\infty} \frac{x+2x^3}{4x^3+4x} + \lim_{x\to\infty} e^{1/x} + \lim_{x\to\infty} \sqrt{3+\frac{x+1}{x}}$$

$$= \frac{1}{2} + 1 + 2 = \boxed{\frac{7}{2}}$$

($e^{1/x} \to 1$ because $1/x \to 0$ as $x \to \infty$)

5. A principal invested now and half that amount invested one year from now will be worth a total of \$4,000 three years from now. Interest is 3.6% APR compounding quarterly. What is the principal rounded up to the nearest dollar? $r = 3.6/(100)(4) = .009$

(A) \$767 (B) \$1,227 (C) \$2,348 $\boxed{\text{(D) \$2,424}}$ (E) none of (A) - (D)

Solve for P where

$$4000 = P(1.009)^{12} + \frac{1}{2}P(1.009)^{8}$$

$\boxed{P \approx 2{,}424}$ (Calculator work)

6. If the equation $x^3y + 3ln(y) + 4\sqrt{x} = 72$ defines y implicitly as a function of x, then y' evaluated at $(4, 1)$ is

$\boxed{\text{(A) } -\frac{49}{67}}$ (B) $-\frac{49}{51}$ (C) $-\frac{1}{49}$ (D) $-\frac{13}{16}$ (E) none of (A) - (D)

Diff implicitly : $3x^2y + x^3y' + \frac{3}{y}y' + \frac{4}{2\sqrt{x}} = 0$

Let $x=4$, $y=1$: $48 + 64y' + 3y' + 1 = 0$

$67y' = -49$

$y' = -\frac{49}{67}$

$$\boxed{\therefore\ y'\Big|_{\substack{x=4\\y=1}} = -\frac{49}{67}}$$

7. If $u = \dfrac{3}{x^2-4}$ then $\dfrac{du}{dx}$ equals (A) $-\frac{2}{3}ux$ (B) $-\frac{u^2x}{3}$ (C) $-2u^2x$ $\boxed{\text{(D) } -\frac{2}{3}u^2x}$

$u = 3(x^2-4)^{-1}$

$u' = -3(x^2-4)^{-2}(2x)$

$= -\frac{2}{3}\left(\frac{3}{x^2-4}\right)^2 x = \boxed{-\frac{2}{3}u^2x}$

***** Make sure your answers are printed in the letter boxes at the top of page 2*****

Part B (Full Solution Questions) Show all of your work. Answers/solutions will earn full points only if they are correct, complete, and sufficiently display relevant concepts from MATA32F.

1. (a) Let $f(x) = xe^{-x^2}$. Find $f'(x)$ and simplify your answer. [4 points]

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x)$$
$$= e^{-x^2}\left[1 - 2x^2\right]$$
$$= e^{-x^2}(1-\sqrt{2}x)(1+\sqrt{2}x)$$

Either is ok.

(b) Let $y = x^3 log_2(x)$. Find $\dfrac{dy}{dx}$ and simplify. [4 points]

Either is ok.

$$\frac{dy}{dx} = 3x^2 \log_2(x) + x^3 \frac{1}{\ln(2)\, x}$$
$$= x^2\left(3\log_2(x) + \frac{1}{\ln(2)}\right) = \frac{x^2}{\ln(2)}\left(3\ln(x)+1\right)$$

$$\boxed{\log_2(x) = \frac{\ln(x)}{\ln(2)}}$$

(c) For $x > 0$ let $h(x) = \begin{cases} \dfrac{8-\sqrt{16x}}{x-4} & if \quad x \neq 4 \\ k & if \quad x = 4 \end{cases}$ [5 points]

Find the value of the constant k that makes h continuous at $x = 4$.

h is continuous at 4 iff $\lim\limits_{x\to 4} h(x) = h(4) = k$

Consider the limit:

$$\lim_{x\to 4} \frac{8-\sqrt{16x}}{x-4}$$
$$= \lim_{x\to 4} \frac{8-4\sqrt{x}}{x-4}$$
$$= \lim_{x\to 4} \frac{4(2-\sqrt{x})}{(\sqrt{x}-2)(\sqrt{x}+2)}$$
$$= 4 \cdot \lim_{x\to 4} \frac{-1}{\sqrt{x}+2}$$
$$= 4 \cdot \frac{-1}{4}$$
$$= -1$$

$$\boxed{\therefore\ k = -1}$$

2. A total debt of \$4,000 due three years from now and \$5,000 due 50 months from now is to be repaid by three payments as follows:

   (i) a first payment is made 8 months from now;

   (ii) a second payment of \$3,000 is made two years from now;

   (iii) a third payment (which is 25% less than the first payment) is made $3\frac{1}{2}$ years from now.

   42 months

   ||

   Interest is 2.4% APR compounding monthly. Find the payments in (i) and (iii) rounded-up to the nearest dollar. A complete money-time diagram and equation of value are required for full points. [11 points]

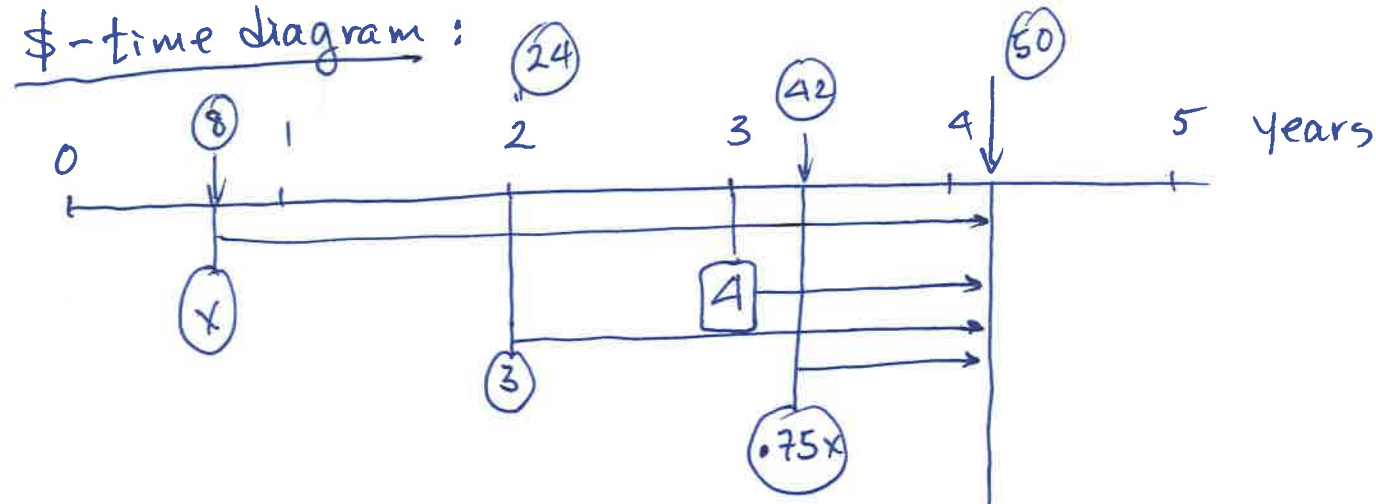
Let x = 1ST payment in (i) above

$\therefore$ payment in (iii) is $0.75x$ $\quad r = \dfrac{2.4}{(100)(12)} = .002$

Assume all money is in units of \$1,000.

Debts : $\square$ Payment : $\bigcirc$

$-time diagram :



At time = 50 months,

Value of all Pay = Value of all Debt

$$x(1.002)^{42} + 3(1.002)^{26} + (0.75x)(1.002)^{8} = 5 + 4(1.002)^{14}$$

Solving gives $x = 3.21876827$ (Calculator work)

Round up to nearest dollar

$\therefore$ $\boxed{\begin{array}{l}\text{Payment (i)} = \$3,219 \\ \text{Payment (iii)} = \$2,415\end{array}}$

3. In all of this question let $p = \sqrt{4,900 - q^2}$ be a demand function where $0 \leq q \leq 70$.

(a) Show that the (point) elasticity of demand satisfies the equation $\eta + \frac{p^2}{q^2} = 0$.

[5 points]

$p^2 = 4,900 - q^2 \qquad p' = \frac{1}{2}(4900 - q^2)^{-1/2}(-2q)$

$= \frac{-q}{\sqrt{4900 - q^2}} = \frac{-q}{p}$

$\therefore \eta + \frac{p^2}{q^2} = \frac{\frac{p}{q}}{p'} + \frac{p^2}{q^2} = \frac{p}{q}\left(-\frac{p}{q}\right) + \frac{p^2}{q^2}$

$= -\frac{p^2}{q^2} + \frac{p^2}{q^2} = 0$

(Also, $\eta = -\frac{p^2}{q^2}$)

(b) Is the (point) elasticity of demand elastic or inelastic when $q = 42$?

[3 points]

When $q = 42$, $p^2 = 4900 - (42)^2 = 3,136$

$\therefore |\eta| = \left| -\frac{p^2}{q^2} \right| = \frac{p^2}{q^2} = \frac{3,136}{1,764} > 1$

$\therefore$ demand is elastic.

(c) Find the value (or values) of q for which the (point) elasticity of demand is unit.

[4 points]

We want $|\eta| = 1$

$\because \eta = -\frac{p^2}{q^2}$, $1 = |\eta| = \frac{p^2}{q^2}$

$\therefore$ We need $4,900 - q^2 = q^2$

$2q^2 = 4,900$

$q^2 = \frac{4,900}{2} \Rightarrow q = \frac{70}{\sqrt{2}} = \boxed{35\sqrt{2}}$

$\boxed{\text{or } 49.497}$

Either is ok.

(We have $q \geq 0$)

4. The two parts of this question are independent of each other.

(a) A car is purchased for $8,995 down now plus bi-weekly payments of $276 for four years. Interest is 5.2% APR compounding bi-weekly and all payments are made at the beginning of each bi-weekly compounding period. Calculate the "cash price" of the car rounded up to the nearest dollar. [7 points]
("Cash price" = full price now, "bi-weekly" = every other week, 1 year = 52 weeks)

$r = \frac{5.2}{(100)(26)} = .002$

26 payments per year
4 years $\Rightarrow 4 \times 26 = 104$ payments

We want PV of annuity due

$$\text{Cash Price} = 8,995 + 276\left[\frac{1-(1.002)^{-103}}{.002}\right] + 276$$

$$= 34,938.928$$

$$\boxed{\therefore \text{ Cash Price} = \$34,939}$$

(b) One of your MATA32F professors claims that an investment at 7.3% APR can yield a 25% return on that investment at the end of three years. Do you believe them? Justify your answer with an appropriate calculation. [5 points]

Consider an investment of $\$P > 0$ @ 7.3% APR. Maximum possible return is when interest compounds continuously.
Suppose this happens.

$$\therefore S_{MAX} = Pe^{(.073)(3)} = Pe^{.219} \approx 1.24483P < 1.25P$$

i.e. Max possible % return is $< 25\%$

$\boxed{\therefore \text{ we do not believe that professor!}}$

5. In all of this question let $g(x) = \sqrt{x^2+32}$. $= (x^2+32)^{1/2}$

(a) Find $g'(x)$ and simplify. [3 points]

$$g'(x) = \frac{1}{2}(x^2+32)^{-1/2}(2x) = \boxed{\frac{x}{\sqrt{x^2+32}}}$$

(b) Find the point (or points) (x, y) on the curve $y = g(x)$ at which the tangent line is parallel to the line $3x + 9y = 5$. [6 points]

$9y = -3x - 5$

$\therefore y = -\frac{1}{3}x - 5$

We solve $g'(x) = -\frac{1}{3}$

$$\frac{x}{\sqrt{x^2+32}} = -\frac{1}{3} \quad (*)$$

$-3x = \sqrt{x^2+32}$

$9x^2 = x^2 + 32 \Rightarrow x^2 = 4$

$(*)$ forces $x < 0$, so we have only $x = -2$

$g(-2) = \sqrt{4+32} = \sqrt{36} = 6$

$\therefore$ the only point is $\boxed{(-2, 6)}$

(c) Use the definition of derivative (i.e. first principles) to calculate $g'(4)$. [6 points]

$$g'(4) = \lim_{x\to 4}\left[\frac{g(x) - g(4)}{x-4}\right]$$

$$= \lim_{x\to 4}\left[\frac{\sqrt{x^2+32} - \sqrt{48}}{x-4}\right]$$

$$= \lim_{x\to 4}\left[\frac{(\sqrt{x^2+32} - \sqrt{48})(\sqrt{x^2+32} + \sqrt{48})}{(x-4)(\sqrt{x^2+32} + \sqrt{48})}\right]$$

$$= \lim_{x\to 4}\left[\frac{x^2+32 - 48}{(x-4)(\sqrt{x^2+32} + \sqrt{48})}\right]$$

$$= \lim_{x\to 4}\left[\frac{(x-4)(x+4)}{(x-4)(\sqrt{x^2+32} + \sqrt{48})}\right] = \lim_{x\to 4}\left[\frac{x+4}{\sqrt{x^2+32} + \sqrt{48}}\right]$$

$$= \frac{8}{2\sqrt{48}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\boxed{\therefore g'(4) = \frac{\sqrt{3}}{3}}$$

Either ok.

Rough: $\frac{8}{2\sqrt{48}} = \frac{8}{(2)(4)\sqrt{3}} = \frac{1}{\sqrt{3}}$

6. Assume the following throughout all of this question: a manufacturer finds that, when 8 units of a product are made, the total cost of production is \$1,112 and \$38.80 is the marginal cost.

(a) Find the approximate cost to produce 9 units. Round your answer up to the nearest 1/100-th of a dollar. [4 points]

Marginal concept for derivative gives

$$C'(8) \approx C(9) - C(8)$$

$$\therefore\ C(9) = C'(8) + C(8) = 38.80 + 1{,}112$$
$$= \boxed{1{,}150.80}$$

In the remainder of this question, assume the cost function C for the product in part (a) and above has the special form $C(q) = Aq + \dfrac{B}{q+2} + K$ where A, B, and K are positive constants and $q \geq 0$ is the quantity produced.

(b) If $\overline{C}(q)$ is the average cost and $\lim_{q\to\infty} \overline{C}(q) = 40$, find A. [3 points]

$$\overline{C}(q) = \frac{C(q)}{q} = A + \frac{B}{q(q+2)} + \frac{K}{q},\quad q > 0$$

$$\therefore\ A = \lim_{q\to\infty} \overline{C}(q) = 40 \qquad \boxed{\therefore\ A = 40}$$

(c) Find the value of the constants B and K. [7 points]

From (b), $C(q) = 40q + \dfrac{B}{q+2} + K$

$$\therefore\ C'(q) = 40 - \frac{B}{(q+2)^2}$$

$$C'(8) = 40 - \frac{B}{100} = 38.80 \qquad \boxed{\therefore\ B = 120}$$

$$\therefore\ C(q) = 40q + \frac{120}{q+2} + K$$

$$C(8) = 320 + 12 + K = 1{,}112$$

$$\boxed{\therefore\ K = 780}$$

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Some Basic Statistics *

$N = 638$ students wrote the test

$\bar{x}$ = average = 61.8%

%	#'s of students	% of 638
100	0	0
90's	60	9.4
80's	107	16.8
70's	104	16.3
60's	83	13
50's	90	14.1
40's	77	12.1
30's	51	8.0
20's	40	6.3
10's	21	3.8
1's	5	0.8

% of N

\# of students passing (ie $\geq 50\%$) = 444 $\approx$ 69.6%

\# " " $\geq 60\%$ = 354 $\approx$ 55.5%

\# " " $\geq 70\%$ = 271 $\approx$ 42.5%

\# " " $\geq 80\%$ = 167 $\approx$ 26.2%

(all very typical of MATA32F)

* All statistics are calculated before any regrading or any transferring to MATA32Y.

On the issue of regrading, almost every request resulted in about a 2–4% increase. There were about 60 regrade requests.