*** SOLUTIONS ***

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32F - Midterm Test - Calculus for Management I

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Date: October 26, 2012
Duration: 120 minutes
Time: 3:00 pm

Last Name (PRINT): SOLUTIONS

First Name(s) (PRINT): (statistics on p. 11)

Student Number: ______________________

Signature: ______________________

Circle your TA and tutorial number:

Natalia Chernega 15	Dongyuan (Sheldy) Shen 11 12
Fazle Chowdhury 1 3 10 13 23	Aaron Situ 4 5
Rui (Ray) Gao 2 22 24	Huiyi Wang 25
Angha Gupta 9 16	Peishan Wang 19 20 21
Daniel Moghbel 6 14 17	Kevin Yan 7 8

Read these instructions:

1. This test has 11 numbered pages. It is your responsibility to ensure at the start of the test that all 11 pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator that does not perform any: graphing, matrix operations, numerical/symbolic differentiation or integration. If your calculator performs any of these operations, then you can not use it during the test. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are strongly encouraged to write your test in pen or other ink. If any questions are answered in pencil, then your entire test is denied any remarking privilege.

* SOLUTIONS *

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7	8
C	C	B	A	C	D	D	E

Do not write anything in the boxes below.

Info.	**Part A**
2	24

Part B

1	2	3	4	5	6
9	15	11	10	12	17

Total
100

Some formulas:

$S = P(1+r)^n$ $\qquad$ $S = Pe^{rt}$

revenue = price × quantity $\qquad$ profit = revenue - cost

Ordinary: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

Due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. If an average cost function is $\bar{c} = 32 + \frac{6}{q} + ln(q)$ where $q > 0$ is quantity, then the marginal cost when $q = 6$ is (approximately)

   (A) 0 (B) 32.17 (C) 34.79 (D) 6.17 (E) a number not in (A) - (D)

$$C = q \cdot \bar{c} = 32q + 6 + q\ln(q)$$

$$C' = 32 + \ln(q) + 1 \qquad C'(6) = 33 + \ln(6) \approx 34.79$$

2. If interest is 5% APR compounding semi-annually, what amount (rounded up to the nearest dollar) on October 26, 1982 would be worth \$10,000 today?

   (A) \$9,279 (B) \$8,609 (C) \$2,273 (D) \$2,723 (E) \$4,768

$$10{,}000(1.025)^{-60} \approx 2{,}272.835$$

3. What 3-decimal approximate annual interest rate compounded continuously is equivalent to 3% APR compounded every four months?

   (A) 2.909% (B) 2.985% (C) 3.980% (D) 2.895% (E) none of (A) - (D)

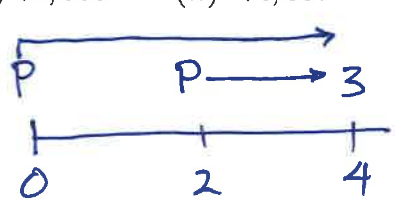
$$e^{r} = \left(1 + \frac{.03}{3}\right)^3$$

$$r = \ln\left[\left(1 + \frac{.03}{3}\right)^3\right] \approx 0.02985099$$

4. An amount P invested now and P invested two years from now will be worth a total of \$3,000 four years from now. If interest is 4% APR compounding semiannually, what is the amount P rounded up to the nearest dollar?

   (A) \$1,331 (B) \$2,366 (C) \$1,414 (D) \$1,668 (E) \$2,667

$$P(1.02)^8 + P(1.02)^4 = 3$$



$$P = \frac{3}{(1.02)^8 + (1.02)^4} \approx 1{,}330.913$$

5. The value of $\lim_{x \to 0} \left(\frac{4x^3 + 2x + 9}{x^3 - x + 3} - e^{(1-x)} \right)$ is (A) 4 (B) 3

(C) $3 - e$ (D) $4 - e$ (E) nonexistent (or $\pm\infty$) (F) a number not in (A) - (D)

Given function is continuous at 0, so we get the limit by substitution : $\frac{9}{3} - e^1 = 3 - e$

6. If $y = 3(2^x) + \sqrt{2x+4}$ then $y'(0)$ equals (A) $\frac{1}{4}$ (B) $\frac{1}{4} + ln(8)$

(C) $\frac{1}{2}$ (D) $\frac{1}{2} + ln(8)$ (E) $\frac{1}{2} + ln(6)$ (F) a number not in (A) - (E)

$$y' = 3(2^x)\ln(2) + \frac{2}{2\sqrt{2x+4}}$$

$$y'(0) = 3\ln(2) + \frac{1}{2} = \frac{1}{2} + \ln(8)$$

7. What is the value of the constant a if $f(x) = \frac{ax^3 + x}{6\sqrt{x}}$ and $f'(1) = 3$?

(A) 1 (B) -1 (C) -7 (D) 7 (E) 3 (F) a number not in (A) - (E)

$$f'(x) = \frac{(3ax^2+1)(6\sqrt{x}) - (ax^3+x)\left(\frac{3}{\sqrt{x}}\right)}{36x}$$

$$3 = f'(1) = \frac{(3a+1)(6) - (a+1)3}{36}$$

$$108 = 18a + 6 - 3a - 3$$
$$= 15a + 3$$
$$105 = 15a$$
$$a = 7$$

8. The least whole number of months it takes an investment to increase by 32% at 3.6% APR compounding eight times annually is

(A) 62 (B) 18 (C) 88 (D) 95 (E) 93 (F) not in (A) - (E)

Let n = the number of compounding periods, $n \in \mathbb{N}$.

Solve for n where

$$1.32 = (1.0045)^n$$

$$n = \frac{\ln(1.32)}{\ln(1.0045)} \approx 61.83$$

Use 62 compoundings

$$1 \text{ compounding} = \frac{12}{8} = 1.5 \text{ months}$$

$\therefore (62)(1.5) = 93$ months

Make sure that your answers are printed in the letter boxes at the top of page 2

Part B (Full Solution Questions) Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32F.

1. A total debt of \$9,000 due four years from now and \$4,000 due 66 months from now is to be repaid with a first payment of \$5,000 now and two more payments as follows:

   (i) a second payment at the end of twenty months from now, and

   (ii) a third payment (which is 70% of the second) made at the end of three years from now.

   Interest is 3% APR compounding monthly. Find the amount of the payments in (i) and (ii) rounded-up to the nearest dollar. [9 points]

   (A correct money-time diagram and equation of value are worth 2 and 3 points, respectively.)

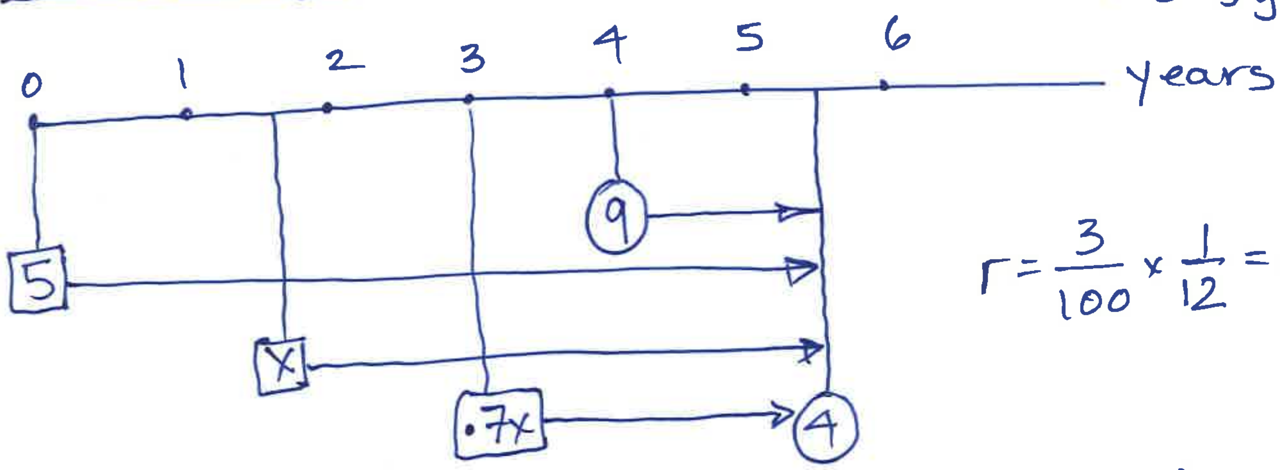
All money will be in 1,000's of \$

Let x represent the amount of 2^{nd} payment in (i)

∴ 0.7x represents payment in (ii) (i.e. 3^{rd})

□ = payment ○ = debt

\$-time diagram with calibration @ "end" (66 months) = 5.5 years



$r = \frac{3}{100} \times \frac{1}{12} = .0025$

Equation of value : Value of all □ = Value of all ○ (@ all time)

$$5(1.0025)^{66} + x(1.0025)^{46} + (0.7x)(1.0025)^{30} = 9(1.0025)^{18} + 4$$

$$\therefore 1.87616073 9x = 7.51797059 1$$

$$x \cong 4.007103677$$

ANSWERS :

Payment (i) = \$4,008

Payment (ii) = \$2,806

2. In all of this question let $f(x) = \dfrac{x+8}{1-x}$

(a) Differentiate f and simplify your answer. [3 points]

$$f'(x) = \frac{1(1-x) - (x+8)(-1)}{(1-x)^2} = \frac{1-x+x+9}{(1-x)^2}$$
$$= \frac{9}{(1-x)^2}$$

(b) Find the equation of each tangent line to the graph of $y = f(x)$ that is parallel to the line $9x - y = 10$. Express your answers in slope-intercept form. [6 points]

$9x - y = 10 \rightarrow y = 9x - 10 \rightarrow$ Slope $= 9$

$\therefore$ slope of desired line $= 9$

Solve for x where $f'(x) = 9$

$$\frac{9}{(1-x)^2} = 9 \Rightarrow (1-x)^2 = 1$$

$\therefore 1 - x = \pm 1$

$\therefore x = 0$ or 2

$f(0) = 8$

Eq$^{\underline{n}}$ is $\boxed{y = 9x + 8}$

$f(2) = -10$

Eq$^{\underline{n}}$ is $\boxed{y = 9x - 28}$

$(y + 10 = 9(x-2))$

(c) Use the definition of derivative to find $f'(x)$. [6 points]

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{x+h+8}{1-x-h} - \frac{x+8}{1-x}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{(x+h+8)(1-x) - (x+8)(1-x-h)}{h(1-x-h)(1-x)}\right]$$

$$= \lim_{h \to 0}\left[\frac{x - x^2 + h - xh + 8 - 8x - x + xh + x^2 - 8 + 8x + 8h}{h(1-x-h)(1-x)}\right]$$

$$= \lim_{h \to 0}\left[\frac{9h}{h(1-x-h)(1-x)}\right] = \lim_{h \to 0}\left[\frac{9}{(1-x-h)(1-x)}\right]$$

$$= \frac{9}{(1-x)^2}$$

3. (a) Let k be a constant. For $0 < x \leq 2$, let $h(x) = k^2(x-1) + kx$ and when $x > 2$, let $h(x) = \dfrac{x^2-4}{x-2} + 2x$. Use the definition of continuity to find the values of k that make $h(x)$ continuous at 2 [6 points]

We require

$$\lim_{x \to 2} h(x) = h(2) = k^2 + k \quad ①$$

$$\lim_{x \to 2^-} h(x) = \lim_{x \to 2^-} \left[k^2(x-1) + kx\right] = k^2 + k$$

$$\therefore \lim_{x \to 2} h(x) = 8 \quad ②$$

$$\lim_{x \to 2^+} h(x) = \lim_{x \to 2^+} \left[\frac{x^2-4}{x-2} + 2x\right] = \lim_{x \to 2^+} \left[x + 2 + 2x\right] = 8$$

Solve $k^2 + k = 8$ by ① and ②

$$k^2 + k - 8 = 0$$

$$\therefore (k+4)(k-2) = 8 \quad \boxed{\therefore\ k = -4 \text{ or } k = 2}$$

(b) Let U represent "utility" as a function of a "budget", B. Assume $U = B^n e^{\alpha B}$ where α and n are positive constants. Show that $BU' = U\big(n + \alpha B\big)$ (prime means derivative of U with respect to B).

[5 points]

$$U' = nB^{n-1}e^{\alpha B} + B^n e^{\alpha B}\alpha$$

$$\therefore BU' = nB^n e^{\alpha B} + B^n e^{\alpha B}\alpha B$$

$$= nU + U\alpha B$$

$$= U(n + \alpha B) \text{ as required.}$$

4. (a) The beneficiary of an insurance policy has the option of receiving a lump-sum payment of \$500,000 now or equal monthly payments for ten years, where all payments are made at the beginning of each month starting now. If interest is 2.4% APR compounding monthly, what is the monthly payment amount rounded up to the nearest dollar?

[5 points]

The \$500,000 payment now represents the PV of the annuity due, where payments are represented by R.

$$r = \frac{2.4}{100} \times \frac{1}{12} = .002$$

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \ldots & 119 & 120 \\ R & R & R & R & \ldots & R & \end{array}$$

$$\underbrace{R \; R \; R \ldots R}_{119 \text{ of these}}$$

Solve for R where

$$500{,}000 = R\left[\frac{1-(1.002)^{-119}}{.002}\right] + R$$

$$\approx 106.8049721\, R$$

$$R \approx 4{,}681.43$$

$\boxed{\therefore \text{Payment is } \$4{,}682}$

(b) Find the nominal rate of interest compounding daily that is equivalent to an effective rate of 3.45% Use 365 days equals 1 year and express your answer as a percentage rounded up to three decimals. [5 points]

Let r represent the nominal rate (i.e. r is the APR)

Solve for r where

$$1 + .0345 = \left(1 + \frac{r}{365}\right)^{365}$$

$$r = \left[(1.0345)^{1/365} - 1\right](365)$$

$$\approx 0.033919793$$

$\boxed{\therefore \text{Nominal rate is } 3.392\%}$

5. Evaluate these limits. Show appropriate justification in your answers.

(a) $\lim\limits_{x\to-\infty}\left[\left(\frac{12x^2-2x^3}{4x^3+3x^2}\right)^3+3\left(8-\frac{1}{x}\right)^{-1}\right]$ $= \left(-\frac{1}{2}\right)^3 + \frac{3}{8} = \boxed{\frac{1}{4}}.$ [4 points]

$\frac{12x^2-2x^3}{4x^3+3x^2} \longrightarrow -\frac{1}{2}$ as $x \longrightarrow -\infty$

$\left(8-\frac{1}{x}\right) \longrightarrow 8$ as $x \longrightarrow -\infty$

(b) $\lim\limits_{u\to 9}\frac{81-u^2}{3-\sqrt{u}}$ [4 points]

$$= \lim_{u\to 9}\frac{(9-u)(9+u)}{3-\sqrt{u}}$$

$$= \lim_{u\to 9}\frac{(3-\sqrt{u})(3+\sqrt{u})(9+u)}{(3-\sqrt{u})}$$

$$= \lim_{u\to 9}(3+\sqrt{u})(9+u)$$

$$= (6)(18) = \boxed{108}$$

(c) $\lim\limits_{x\to 0}\frac{e^{2x}+x-1}{x}$ (A solution using l'Hopital's rule will earn no points) [4 points]

Let $f(x) = e^{2x}+x$ so $f(0) = 1$ and

$f'(x) = 2e^{2x}+1$ so $f'(0) = 2+1 = 3$

Thus, $\lim\limits_{x\to 0}\frac{e^{2x}+x-1}{x} = \lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}$

$$= f'(0) = \boxed{3}$$

6. In all of this question let $q > 0$ represent number of units of a product (i.e. quantity) sold and let $p = \dfrac{840}{2q+6}$ be the unit price of the product when q units are sold (p is also called the demand function).

(a) Find the marginal revenue when $q = 7$. [5 points]

$$\text{Revenue} = r = pq = \frac{840q}{2q+6}$$

$$\frac{dr}{dq} = \frac{840(2q+6) - 840q(2)}{(2q+6)^2}$$

$$\left.\frac{dr}{dq}\right|_{q=7} = \frac{840(20) - 840(14)}{(20)^2} = \boxed{12.6}$$

(b) Let $c = f(q)$ be a cost function. Assume the marginal cost of 7 units is 4.6 and the average cost of 7 units is 15. Estimate the profit obtained when 8 units are sold. [6 points]

Let $P(q)$ represent profit as a function of q $\therefore P(8) = r(8) - c(8)$

$$r(8) = \frac{840(8)}{22} = 305.\overline{45}$$

$$\bar{c}(7) = 15 \quad \therefore c(7) = 7(15) = 105$$

$$c(8) \approx c(7) + c'(7) = 105 + 4.6 = 109.6$$

$$\boxed{\therefore P(8) \approx 195.85}$$

(c) Show that demand is elastic for all $q > 0$. [6 points]

Elasticity of demand

$$\eta = \frac{\frac{p}{q}}{p'(q)} = \frac{\left[\frac{840}{q(2q+6)}\right]}{\left[\frac{-2(840)}{(2q+6)^2}\right]} = \frac{840}{q(2q+6)} \times \frac{(2q+6)^2}{-2(840)}$$

$$= -\frac{2q+6}{2q} = -\left(1 + \frac{3}{q}\right)$$

$\therefore |\eta| = 1 + \frac{3}{q} > 1$ for all $q > 0$ because $\frac{3}{q} > 0$.

That shows demand is elastic for all $q > 0$.

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Basic Test Statistics

734 students wrote the test

Test average before any regrading is ≈ 64.4%

(average after regrading is about 65%)

Decile percentages before regrading:

100 → 0.14%
90's → 9.8%
80's → 18.1%
70's → 16.1%
60's → 16.6%
50's → 16.6%
40's → 9.4%
30's → 7.0%
20's → 3.1%
10's → 2.7%
1's → 0.4%

About 10% got in 90's (or 100%)

About 28% got ≥ 80%

About 44% got ≥ 70%

About 22.6% got < 50% and about 13.2% got < 40%.

All of these numbers are, historically, very typical of MATA32F.

About 77.4% of all students passed the test.